THE UNIVERSITY OF ALBERTA

SAMPLED-DATA SYSTEM WITH A QUANTIZER

by

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The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies for acceptance, a thesis entitled "Sampled-data System With a Quantizer", submitted by Ephrem Hiu Chung Cheng in partial fulfillment of the requirements for the degree of Master of Science.

SUMMARY

This thesis deals with the investigation of a sampled-data system, with a quantizer in the forward loop.

Two methods of simulating the nonlinear element (quantizer) were successfully developed. Both utilized analog computer components which consisted mainly of comparators and amplifiers.

The transient response of an error-sampled, second-order, unity-feedback, quantized system was studied. Treating the quantizer as a nonlinear element, the state-variable technique was used for the purpose of analysis. Calculated results (by means of a digital computer) and results obtained on the analog computer were in agreement.

A stability criterion by Tsypkin [11], based on the work of Popov, was used to determine the globally asymptotic stability of the system with respect to the gain of the linear plant. All information in the frequency domain was converted into a gain-phase plot, from which the stability criterion could easily be interpreted.

A recently modified criterion by Jury and Lee was also considered and it yielded much less conservative results. In this case, a digital computer was used to apply the criterion.

Another study carried out in the Z-domain indicated that a filter was effective in compensating the system for both cases so that a larger gain was possible; or, in other

words, the relative stability of the system was increased. The filter so found was physically realizable.

ACKNOWLEDGEMENT

The writer wishes to express his gratitude for the advice, encouragement, and assistance which was patiently provided by his supervisor, Professor Y.J. Kingma. The basic ideas of simulation of the quanitzer and compensation for stability of the quantized sampled-data system under study originate from Professor Kingma.

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1. Introduction

In sampled-data control systems, the digital computer is becoming more and more important because of its high accuracy, speed, versatility and flexibility. From the inherent nature of numerical manipulation, the problem of quantization arises naturally. Especially, in dealing with hybrid simulation or computation, in the areas of digital to analog conversion or vice versa, the process requires the unavoidable rounding off of numbers which can be considered as quantization errors. When the accuracy of the analog device is specified, the capacity of the digital device in the sense of word-lengths is determined accordingly in terms of the fineness of quantization. What is more important is the problem of stability, when there is a quantizer in the system.

This thesis deals with the stability of a sampled-data system with a single quantizer in the forward loop. The general approach is to regard the quantizer as a nonlinear, time-invarient, memoryless element, which is, in a sense, a relay with multiple levels of output. Thus, the system has become a nonlinear sampled-data system which, due to lack of completeness and uniqueness of basic principles in this area, causes much difficulty in analysis and synthesis. Moreover, as a result of the difficulty of

describing the quantizer in mathematical terms, the general approach to the analysis of this kind of situation is by graphical methods. There still remains the problem of how to simulate a quantizer and compensate for stability of such systems. In this thesis, a new method for solution of these problems is presented.

2. Analog Simulation of a Quantizer

A quantizer is a nonlinear, time-invariant, memory-less element with an input-output transfer characteristic as shown in figure 1.

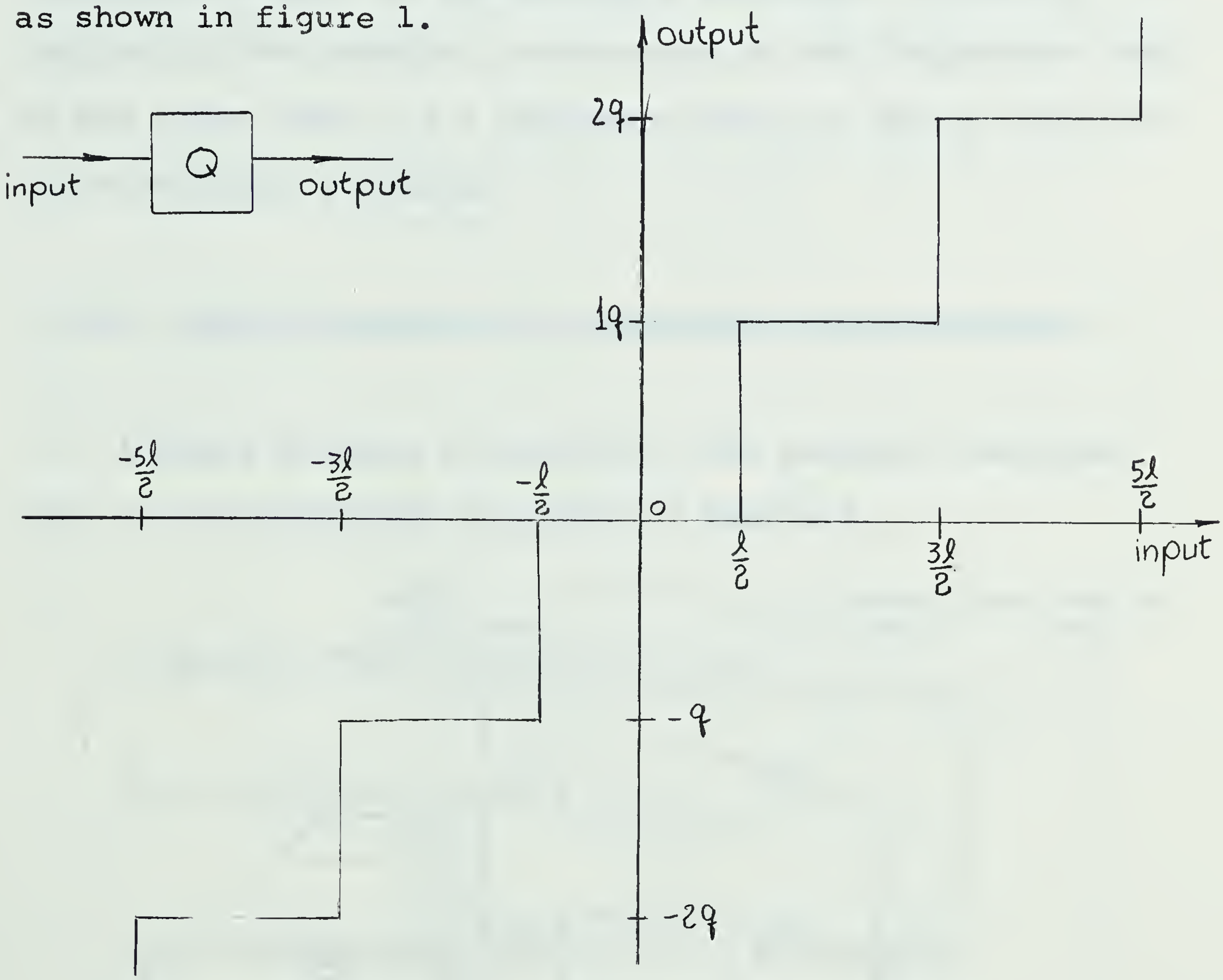


Figure 1. Transfer curve of a quantizer.

At the present, there are several methods available for constructing a quantizer (1),(2),(3). However, none of these is satisfactory because of the following disadvantages; excessively long response-time, complex solid-state circuitry, inaccuracy, drift of output as

time elapses, wear and tear of mechanical parts, and very limited number of steps achieved.

Using the analog computer, a very simple technique was found to be satisfactory. It is referred to as the "Parallel Method" of simulating a quantizer. This is implied by the parallel arrangement of the components used. On the other hand, as a conjugate case, a "Series Method" was also found possible.

2(A) Parallel method for simulating the quantizer

A block diagram illustrating the parallel construction of the quantizer is given in figure 2.

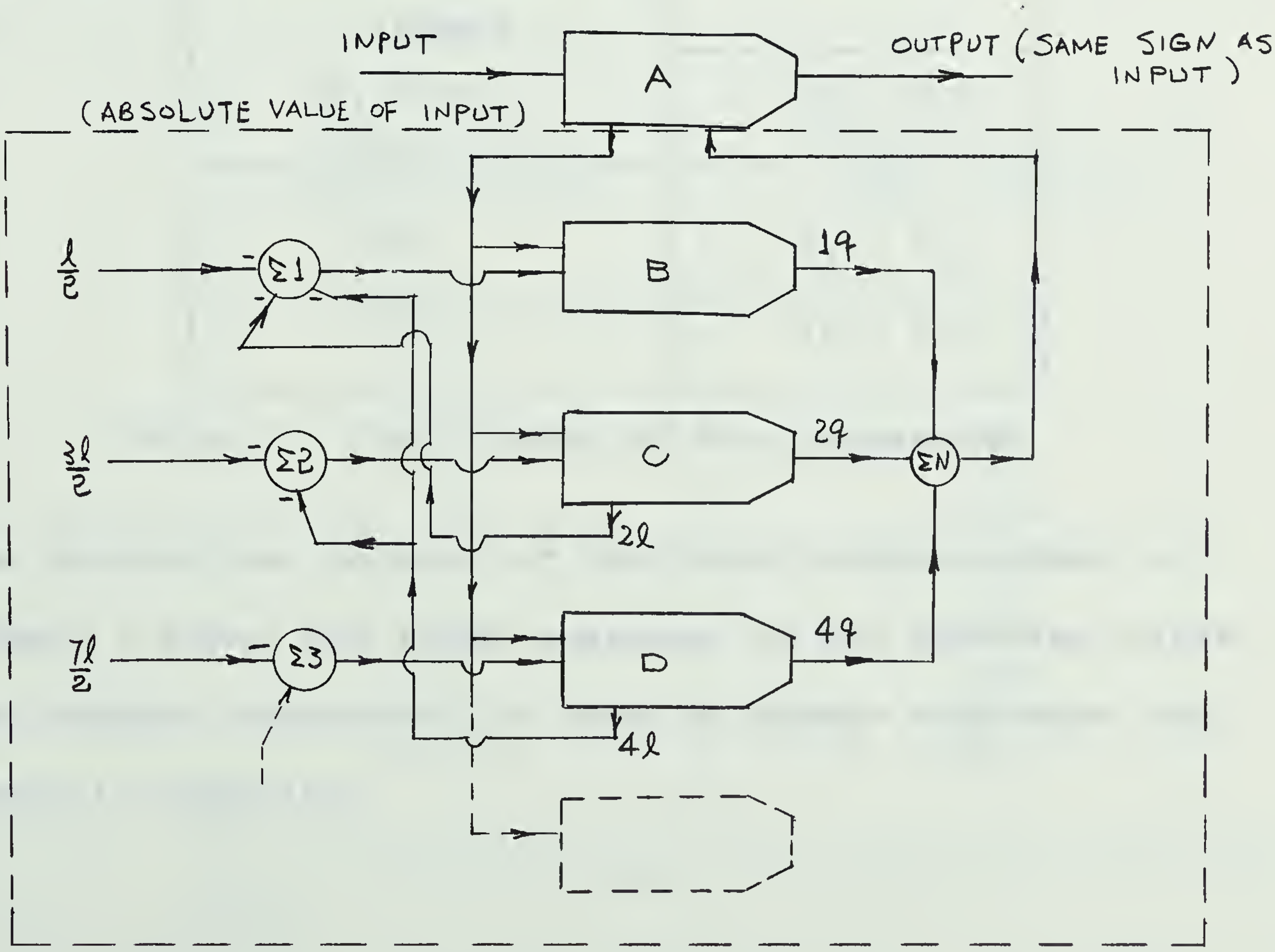


Figure 2. Block diagram showing parallel method of simulating a quantizer.

Comparators, A, B, C, D ..., are illustrated in figure 3.

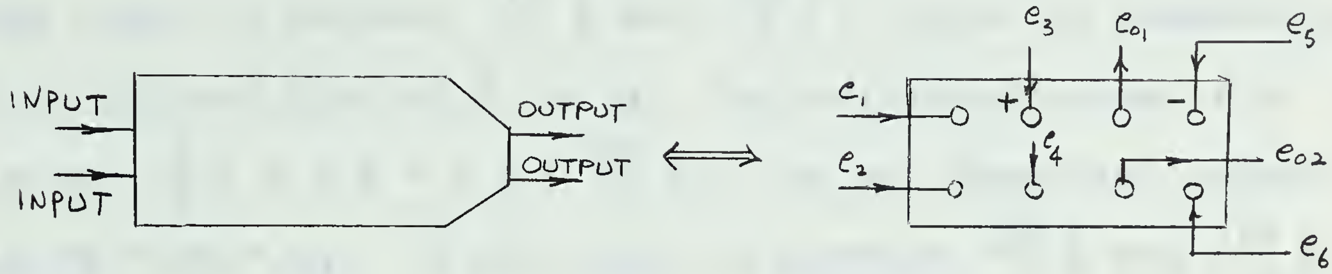


Figure 3. Block diagram of a double-pole-double-throw comparator.

Inputs and outputs are indicated by direction of arrows. The function of the comparator in figure 3 is illustrated by table 1.

Algebraic sum of inputs $e_1 + e_2$	outputs	
	e_{01}	e_{02}
>0	e_3	e_4
<0	e_5	e_6

Table 1. Truth table of the comparator.

The dotted-line portion of the block diagram shown in figure 2 gives the first quadrant of the transfer curve, and another comparator is used to change sign when the input is negative.

Operation of the scheme in figure 2 can be understood better by considering two typical examples as follows. If the input is between $\frac{11}{2}\ell$ and $\frac{13}{2}\ell$, output of comparator D is 4q, and that of C is 2q. The switching value of B is now $\frac{1}{2}\ell + 4\ell + 2\ell = \frac{13}{2}\ell$. The net quantizer output is $4q + 2q = 6q$. If the input is between $\frac{13}{2}\ell$ and $\frac{15}{2}\ell$, outputs of D and C are still respectively 4q and 2q. However, input of the first one, B, can now overcome its switching value of $\frac{13}{2}\ell$, thus giving 1q as its output. The net quantizer output is now $6q + 1q = 7q$. The first comparator, A, is used to yield the same sign for both the input and output. Following the same steps as above, the number of steps possible and the number of comparators used is tabulated as follows:

* Number of Comparators Used	Total Number of Steps Possible
1	2
2	6
3	14
4	30
5	62
6	126
7	254
8	510
9	1022
10	2046
11	4094
12	8190
13	16382
⋮	⋮

Table 2. Relation between number of comparators and number of steps.

* For each case, one more comparator has to be used for sign-inversion.

A schematic diagram of 254 steps ($\pm$ 127 steps), each of which is one volt, is shown in figure 4. For the purpose of convenience, the same scale ($q = \ell = 1$) is used for both the input and output. This scheme requires eight comparators, fifteen amplifiers and fourteen pots. For a computer with one hundred or more amplifiers, ten or more comparators, one hundred or more pots, this does not create any difficulty due to a shortage of components.

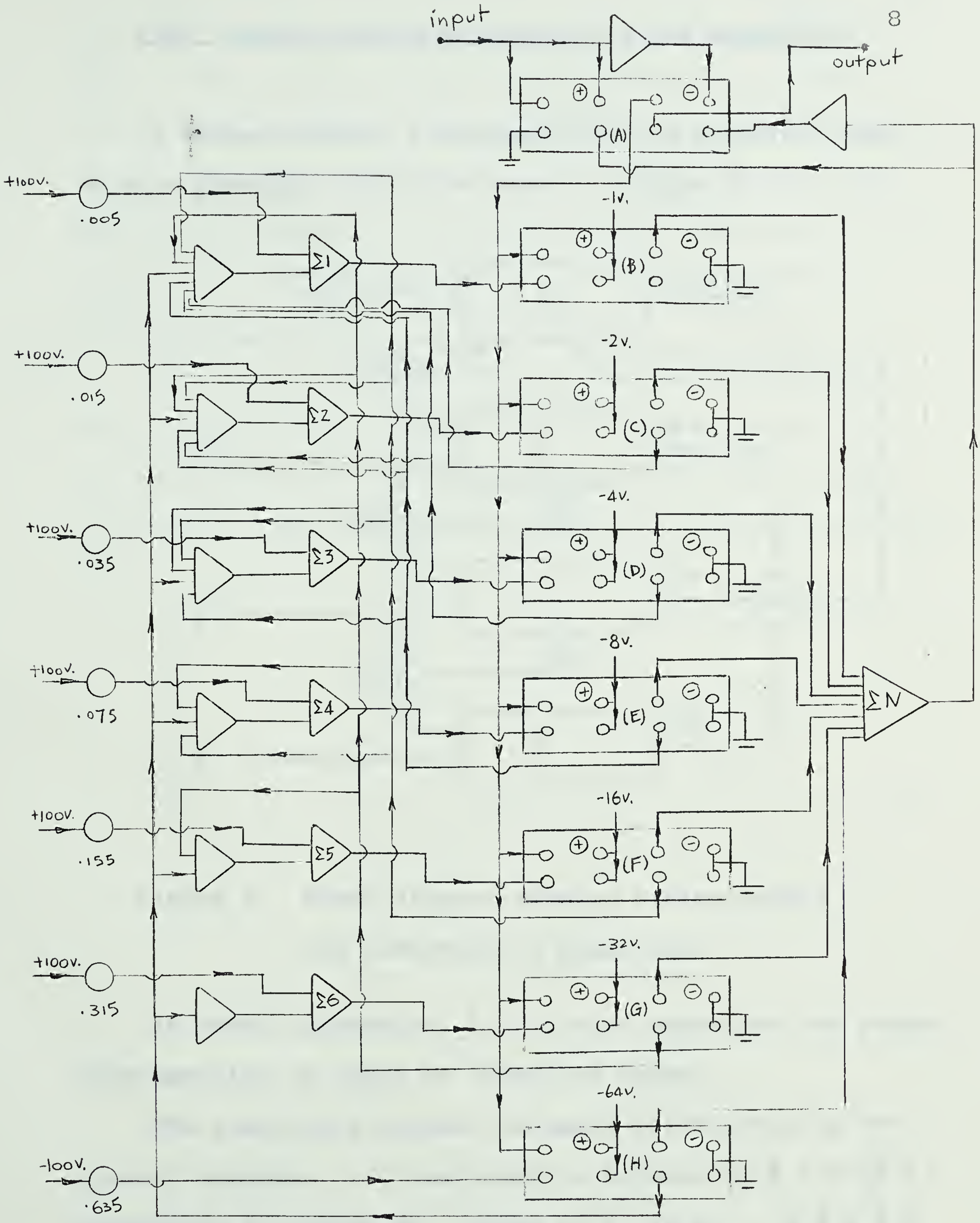


Figure 4. Schematic diagram of parallel method ($q = \ell = 1$).

2(B) Series method of simulating the quantizer

A series method, a conjugate of the parallel type, is also possible. This is shown in figure 5.

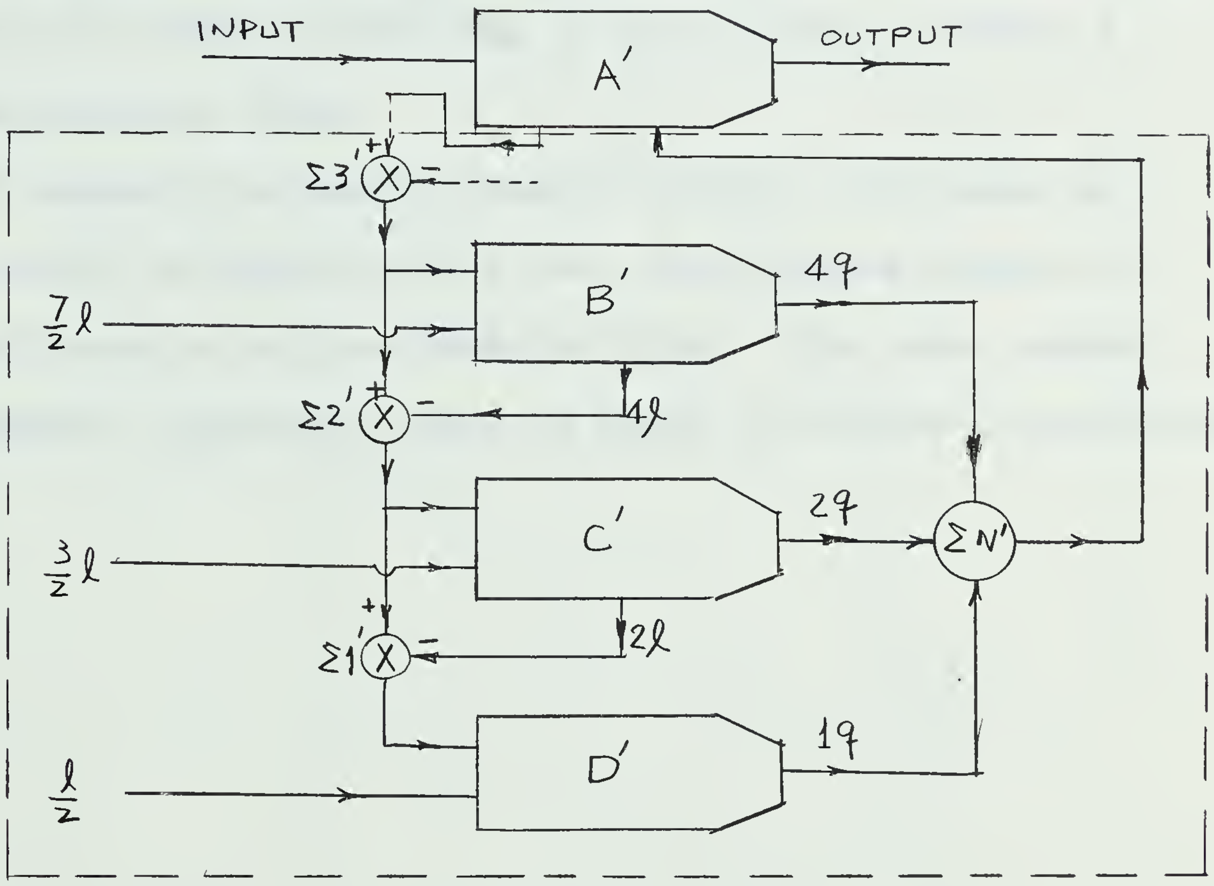


Figure 5. Block diagram showing series method for simulating a quantizer.

An extra comparator, A', is also necessary for proper consideration of signs of input and output.

The quantizing process is again illustrated by two typical examples. If the input is between $\frac{11}{2}\ell$ and $\frac{13}{2}\ell$, comparator B' yields 4q. Input to C' is $e_{in} - 4\ell \geq \frac{3}{2}\ell$ which causes its output to be 2q. Input to D' is $e_{in} - 4\ell - 2\ell \leq \frac{1}{2}\ell$ which is not sufficient to cause any output. The

net result is 6q. If the input is now between $\frac{13}{2}\ell$ and $\frac{15}{2}\ell$, outputs from B' and C' are respectively 4q and 2q. Input to D' is $e_{in} - 4\ell - 2\ell \geqslant \frac{1}{2}\ell$ which causes an output of 1q. The net result is 7q. The relation between the number of comparators necessary and the steps feasible was found to be exactly the same as that given in table 2 of the previous case.

A complete schematic diagram with $\pm$ 127 steps of unity scale is shown in fig. 6. This scheme yields identical results as the parallel type. The total number of computer components used is equal to that of the previous case.

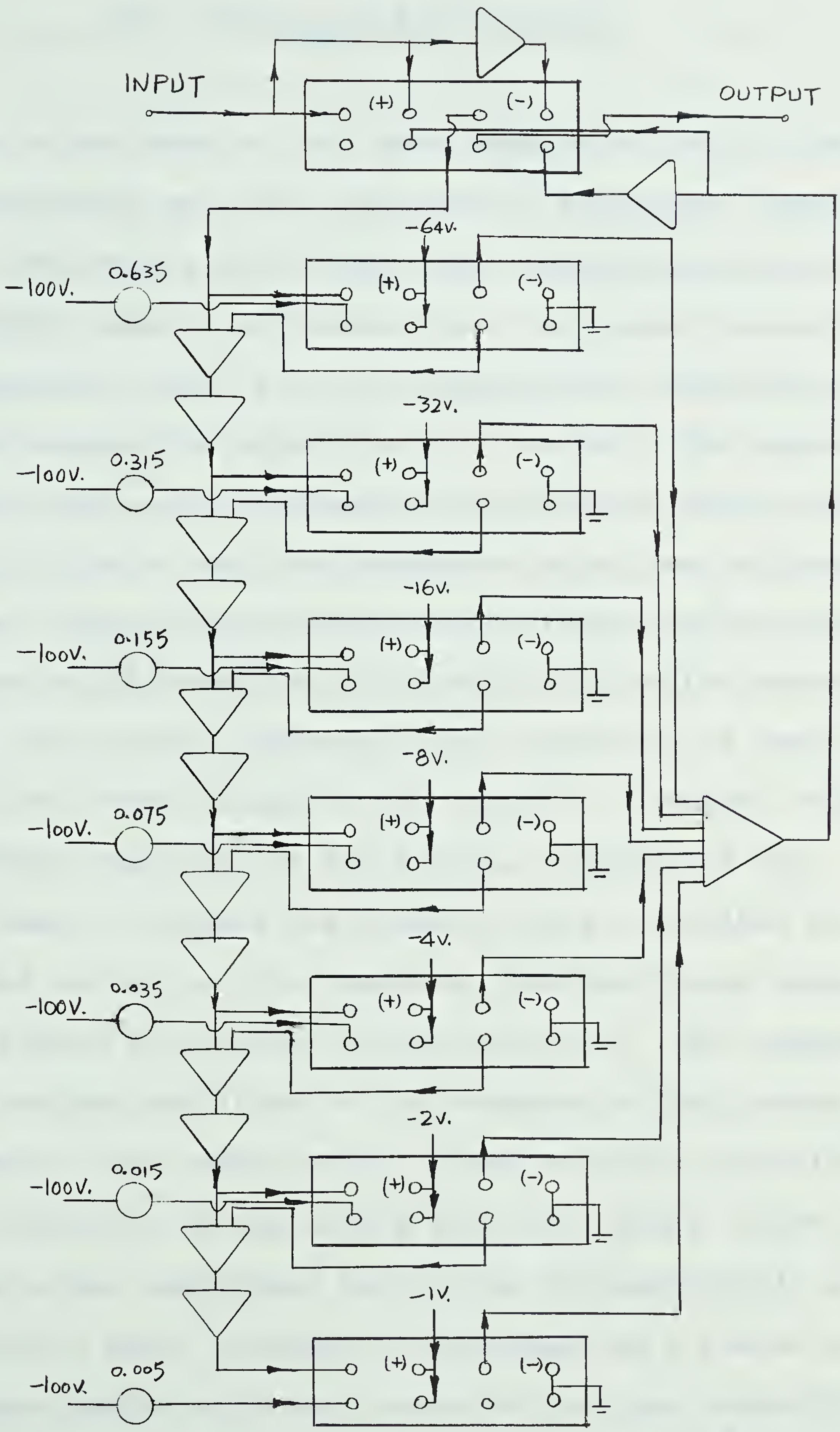


Figure 6. Schematic diagram of series method ($\ell = q = 1$).

2(C) Merits of both methods

As in the case of the ideal sampler or relay, the construction of an ideal quantizer is a physical impossibility. The merits of the quantizer realized by the above two methods depend, of course, to a very great extend on the components used; i.e., the comparators, amplifiers, and the accuracy by which the pots are set. The comparators available are electromechanical devices with a dead zone of $\pm$ 10 m.v. and the response-time of two milliseconds maximum. Pot-setting accuracy as indicated by the digital voltmeter on the computer is also limited to the order of 10 m.v. The actual accuracy of the quantizer is satisfactory for normal usage on the computer. However, when the maximum amplitude of the overall response of the system does not exceed one hundred volts as limited by the rated voltage of the computer, the quantizing steps can be scaled up for even greater accuracy. For example, if the maximum amplitude of the response of the system is not greater than twenty volts, a one-volt-step quantizer can be scaled up to one having five-volt steps; which means that the other amplitudes have to be correspondingly scaled up. In this case, accuracy is increased by a factor of five.

Experiments performed indicated that any correctly quantized level in the output could be reached in less

than approximately five milliseconds. For a sampled-data system with a sampling period of the order of one second or even one-tenth of a second, the duration of five milliseconds is negligible in most experimental work.

A great advantage of these two methods over all others is that the desirable amount of each step can be adjusted very easily; i.e., the values of q and ℓ are changed merely by individual pot-setting. In addition, there is no drift of voltage in the output as normally occurs in analog memory devices. Therefore, the methods discussed are satisfactory in terms of flexibility, large number of steps possible, fast response-time, convenience, simplicity and cost of construction.

3 Transient Response of S.D. System with a Quantizer in the Forward Loop

A second-order, error-sampled, unity-feedback system was chosen for transient response analysis and it is shown in figure 7.

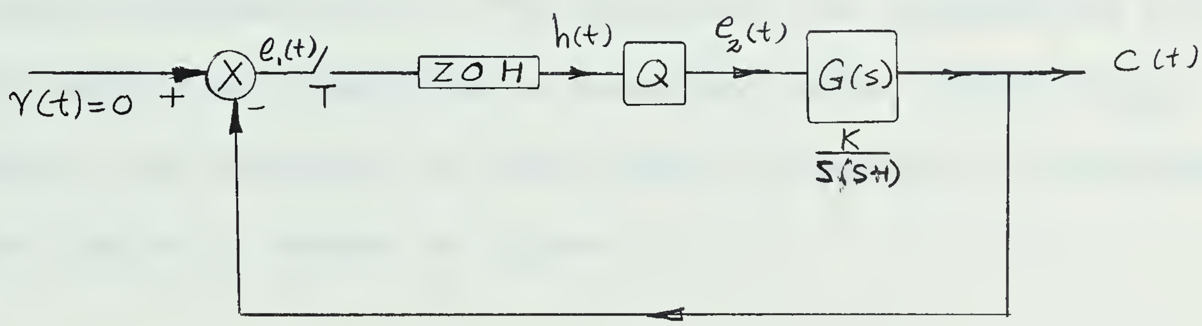
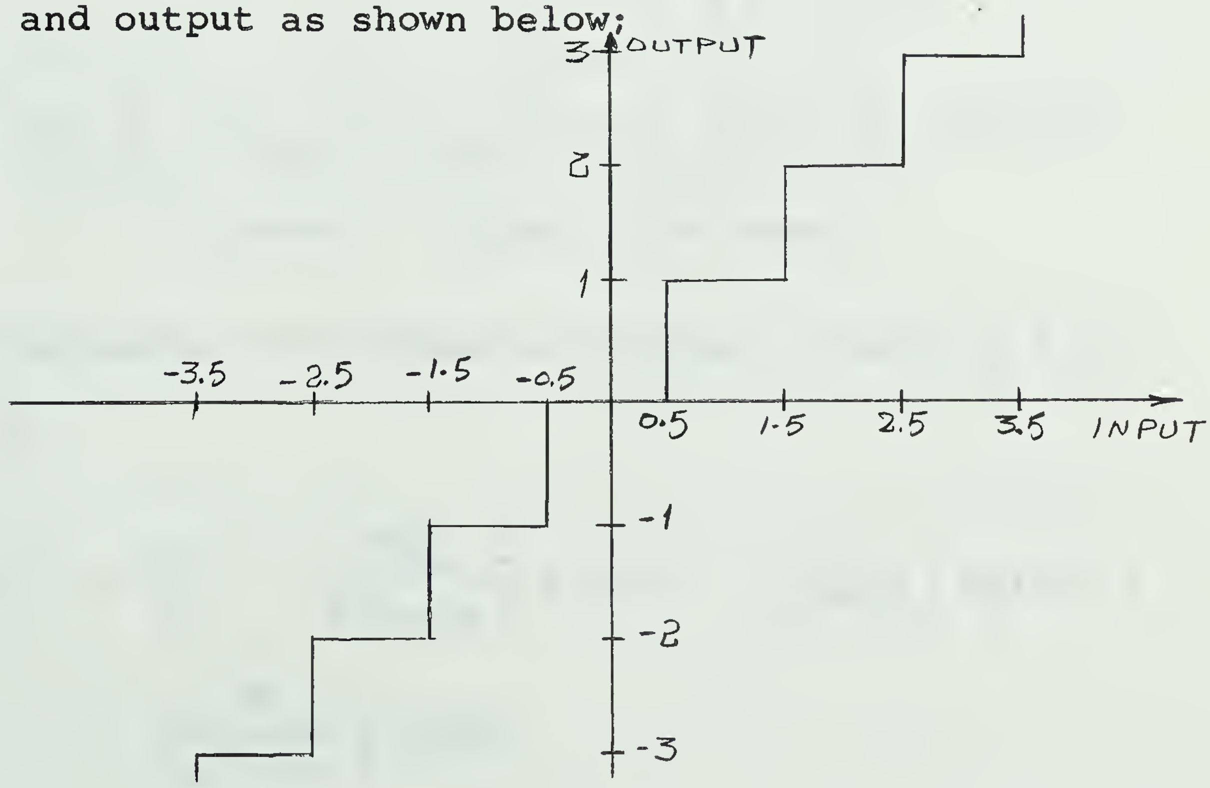


Figure 7. A basic quantized S.D. system.

The unit for the quantizer is unity (ℓ = q = 1) on both input and output as shown below:



In figure 7, r(t) is set to be zero. On the analog computer, experiments were performed with various initial

conditions and gains of the linear plant. Phase-plane diagrams were used to observe the transient response. With different values of gains, stable response or limit-cycle oscillation was observed (the critical value of gain for stability will be discussed later).

The state-variable technique was adopted for theoretical consideration. The quantizer is treated as a variable-gain amplifier which has a constant gain, $K(KT) = K_k$, during the interval of each sampling period. The signal flow graph is shown in figure 8.

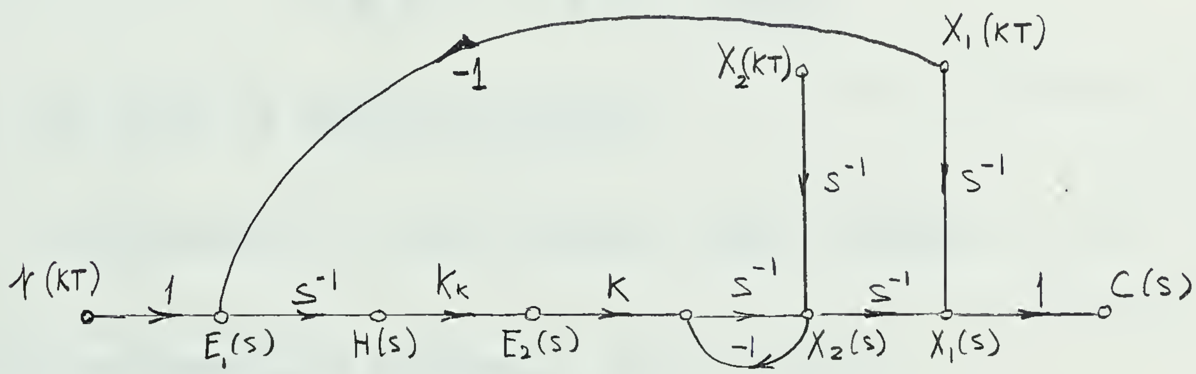


Figure 8. Signal flow graph.

From the signal flow graph in figure 8, where $x_1 = C$, $x_2 = \dot{x}_1$;

$$x_1(s) = \left[\frac{1}{s} - \frac{KK_k}{s^2(s+1)}\right] x_1(KT) + \left[\frac{1}{s(s+1)}\right] x_2(KT) + \left[\frac{KK_k}{s^2(s+1)}\right] r(KT)$$

$$x_2(s) = \left[\frac{-KK_k}{s(s+1)}\right] x_1(KT) + \left[\frac{1}{s+1}\right] x_2(KT) + \left[\frac{KK_k}{s(s+1)}\right] r(KT)$$

taking the inverse-Laplace transform;

$$x_1\left[(k+1)T\right] = \left[1 - KK_k\ (T - 1 + e^{-T})\right] x_1(KT) + (1 - e^{-T})\ x_2(KT)$$

$$+ K_kK\ (T - 1 + e^{-T})\ r(KT)$$

$$x_2\left[(K+1)T\right] = \left[-K_kK\ (1 - e^{-T})\ x_1(KT) + e^{-T}\ x_2(KT)\right.$$

$$+ K_kK\ (1 - e^{-KT})\ r(KT)$$

in the form of matrix algebra;

$$\overline{X}\left[(K+1)T\right] = \overline{A}(T)\ \overline{X}(KT) + \overline{B}(T)\ \overline{R}(KT)$$

for intersampling response, for $0 < \Delta < 1$

$$\overline{X}\left[(K+\Delta)T\right] = \overline{A}(\Delta T)\ \overline{X}(KT) + \overline{B}(\Delta T)\ \overline{R}(KT)$$

In our case, let $T = 1$ sec., $r(t) = 0$, the state-transition equations become;

$$x_1\left[(K+\Delta)T\right] = \left[1 - K_kK(\Delta T - 1 + e^{-\Delta T})\right] x_1(KT) + (1 - e^{-\Delta T})$$

$$x_2(KT)\ \ldots \qquad (1)$$

$$x_2\left[(K+\Delta)T\right] = -K_k K(1 - e^{-\Delta T})\ x_1(KT) + e^{-\Delta T}\ x_2(KT)\ \ldots \qquad (2)$$

For manipulation of the transient response point by point, a digital program was written in FORTRAN 4 and the solution obtained on the IBM 7040. The complete program is given in Appendix 1. The quantizer in the digital programming was formulated by use of the relation between floating-point and fixed-point numbers. The result so obtained checked well with that obtained on the analog computer.

The usefulness of the state-variable technique with the aid of a digital computer is certainly not limited to a second-order system. With systems of higher order and nonunity feedback, the transient response can still be computed step by step in the same fashion.

4 Stability Criterion (Tsypkin's Case)

Based on the work by Popov for the case of a nonlinear continuous system, Tsypkin[11] has derived a criterion for the determination of absolute stability in-the-large for nonlinear sampled-data systems. The basic system is shown in figure 9.

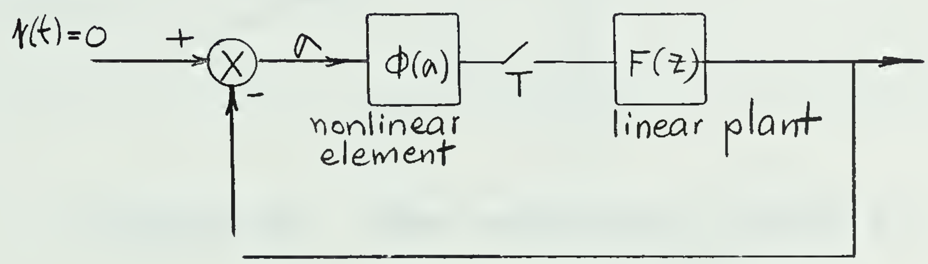


Figure 9. Basic system for Tsypkin's case.

If $\Phi(0) = 0$, $0 < \frac{\Phi(\sigma)}{\sigma} < k$ for $\sigma \neq 0$, and if the linear plant $F(Z)$ is stable; then the autonomous system is globally asymptotically stable provided that the following condition is satisfied:

$$\frac{1}{k} + \operatorname{Re} F(Z) \geq 0 \text{ for every value of } Z \text{ on the}$$

unit circle $|Z| = 1$; where k is the sector to which the nonlinear characteristic is confined as shown in figure 10.

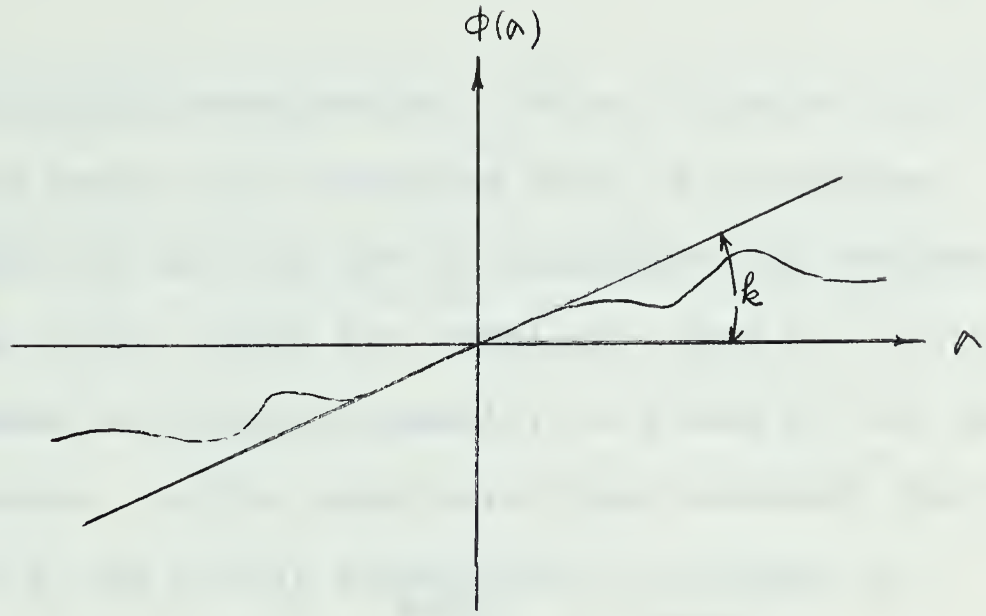


Figure 10. The nonlinear sector k.

The condition that requires $0 < \frac{\phi(\sigma)}{\sigma} < k$ for $\sigma \neq 0$ can be modified if a quantizer is considered to be the nonlinear element, since it does not hold for the range where $|\sigma| < \frac{1}{2}\ell$. This can be done by allowing the steady-state error to be within the range of $\pm \frac{1}{2}\ell$ instead of zero(11). The result is the same since the output of the quantizer is zero within this range.

The question of absolute stability has thus become considerably easier as it refers only to the frequency response of the linear plant against the region of confinement of the nonlinearity; and this is, as the word "absolute" implies, independent of the shape of the transfer curve of the nonlinearity. However, this is a sufficient but not necessary condition, since the result so

obtained is quite conservative. This criterion will be investigated before the modified form is considered.

A study was carried out to determine the maximum gain of the linear plant for stability once the sector of confinement of the nonlinearity is fixed by the specific element chosen. In the quantizing case studied, the sector, k, is fixed to be two as shown below in figure 11.

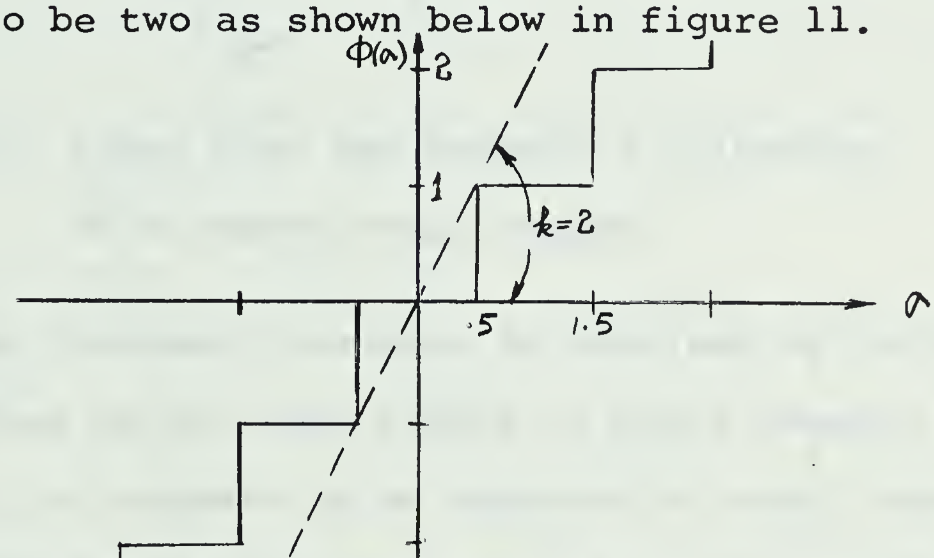


Figure 11. k = 2 for the quantizer under study.

The idea of frequency response implies the commonly used polar diagram. Since the maximum gain of the plant is required, a minimum value of the real part of the frequency response is expected when the gain is set at unity; or

$$-\frac{1}{k} \leqslant \min \operatorname{Re}\left[F(Z)\right] \quad \ldots\ldots\ldots \qquad (4)$$

for every point on $|Z| = 1$.

The above condition yields the polar diagram of a second-

order system shown in figure 12.

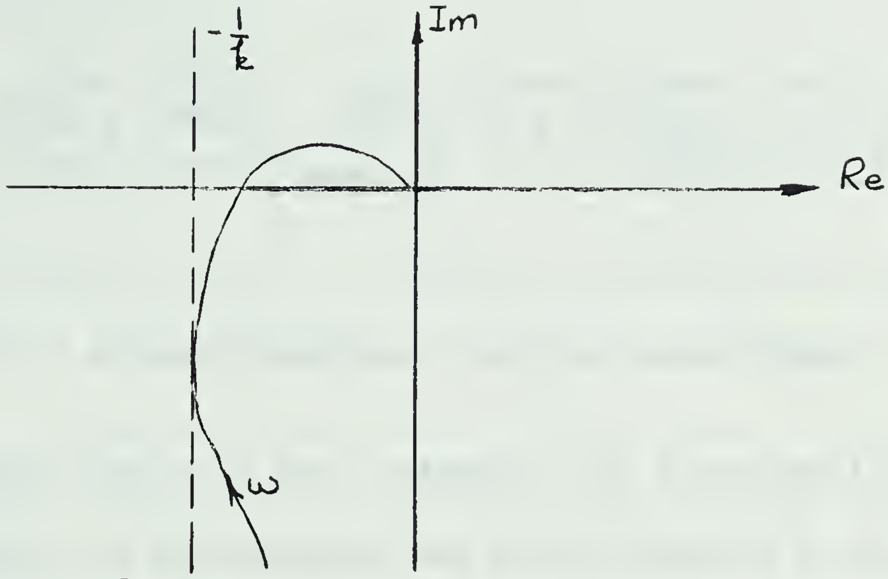


Figure 12. Polar plot for Tsypkin's criterion of a second-order system.

Since the frequency response is obtained by having Z take on values on the unit circle in the Z-domain; and the frequency, ω, appears as an exponent in every term; there is no quick and systematic way to obtain the response as could be done with a Bode plot in the continuous case. The tedious job could be handled very easily by a digital computer, and the minimum real part obtained directly. A new graphical method is also possible.

A commonly used system, error-sampled and followed by a zero-order-hold before the nonlinearity is certainly not of a form to which Tsypkin's criterion can be directly applied. The difficulty could be solved by interchanging positions of the appropriate elements in the system as

shown in figure 13.

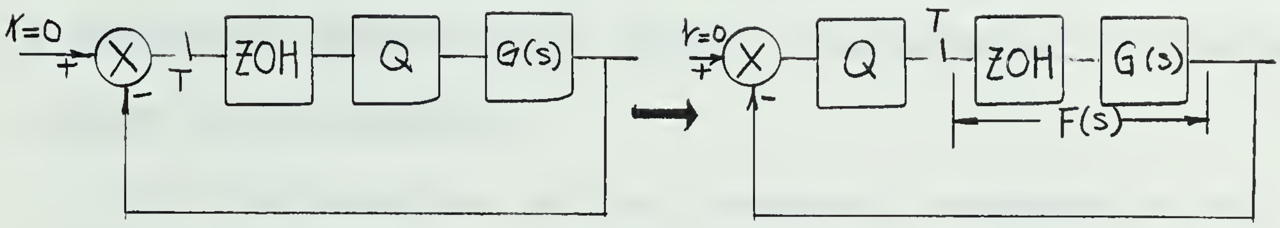


Figure 13. A new location for the quantizer.

It can be shown that the two schemes are identical as far as input to G(s) is concerned, so that Tsypkin's criterion can be directly applied.

The linear plant is now F(s)

$$= \frac{1 - e^{-sT}}{s} \; G(s) \; = \frac{1 - e^{-sT}}{s} \; \frac{K}{s(s+1)} \qquad \text{and}$$

$$F(Z) = \mathcal{Z}\left[F(s)\right]$$

$$= \frac{e^{-1}(Z+p)}{(Z-1)(Z-e^{-1})} \qquad \text{for } T = 1 \text{ second}, \quad K = 1,$$

$$\text{where} \quad p = \frac{(1 - 2e^{-1})}{e^{-1}} \approx 0.717.$$

For this case, results obtained on the digital computer indicated that the minimum real part was -1.5 [the analytical result is $-(\frac{T}{2} + 1)$.]. For k = 2, and substituting the numerical values into equation 4 results in

$$-\frac{1}{2} \leqslant -1.5K \qquad \text{or} \qquad K \leqslant \frac{1}{3}, \qquad \text{and}$$

the maximum gain is found to be 1/3.

In addition to the above method, a graphical technique is also possible for interpreting the criterion in a different manner which enables the problem of compensation to be handled.

All information in the frequency response is now converted into a gain-phase plot with the amplitude in decibels for convenience, and the phase angle in degrees. The $\frac{1}{k}$ line which becomes a curve in the gain-phase plot will be referred to as the "Tsypkin curve" and is shown in figure 14. In this case, for the condition to be satisfied, the Tsypkin curve has to be above or just tangent to the curve plotted for the linear plant for all frequencies. Therefore, the Tsypkin curve is plotted on transparent paper; and, in order to maintain the same phase relation, it is shifted up and down on the gain-phase plot until the Tsypkin curve is tangent to that of the linear plant. The intersection of the Tsypkin curve with the vertical axis indicates the minimum value of the real part of the frequency response of F(Z).

For example, the second-order system shown in figure 13 is considered again. The frequency response of such a system is plotted in figure 15 in which the Tsypkin curve is superimposed. The intersection, point A, gives a value of approximately 3.75 db or 1.54 (actual value

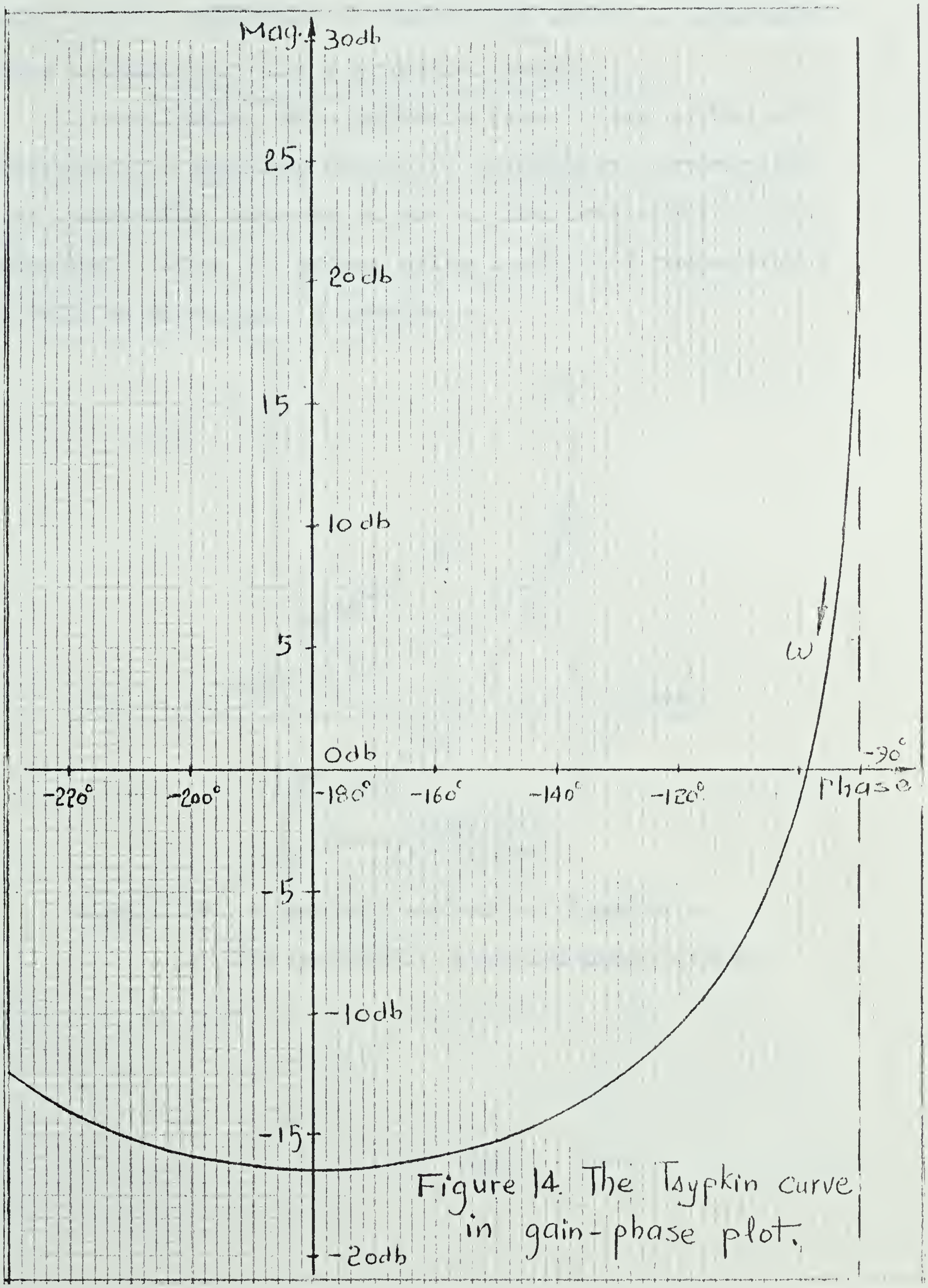


Figure 14. The Tsypkin curve in gain-phase plot.

being 1.50). The error is about 2.6% which is considered quite satisfactory for a graphical method.

In conclusion, this approach gives a new method of obtaining the required stability information graphically with reasonable accuracy as far as absolute stability is concerned. Also, it proves quite useful for compensation as will be discussed in section 6.

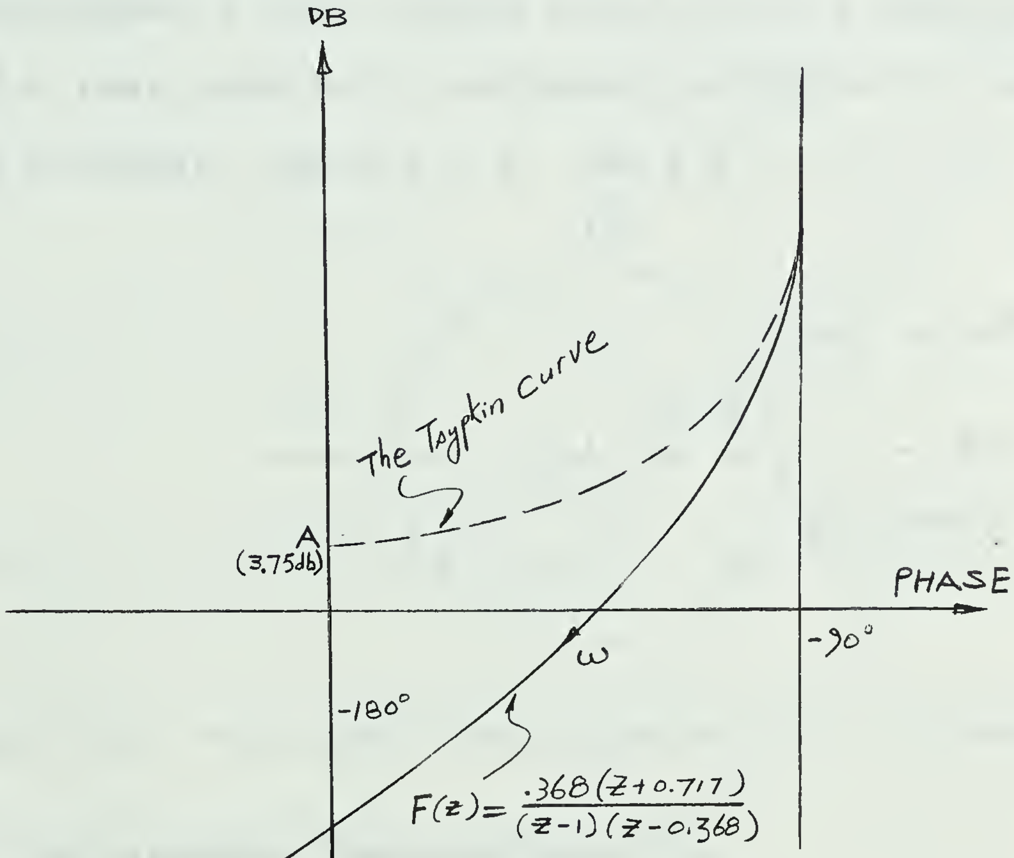


Figure 15. Graphical method for Tsypkin's criterion of a second-order system.

5 A Phase-lead Filter in the Z-domain

The properties of a phase-lead filter will be discussed and the conditions sufficient for compensation will be derived to facilitate compensation in the Z-domain.

To begin, a very simple filter with a real pole at B, and a real zero at A, as shown in figure 16, is assumed in the Z-domain; where $B > A$ and $B < 1$, $A < 1$.

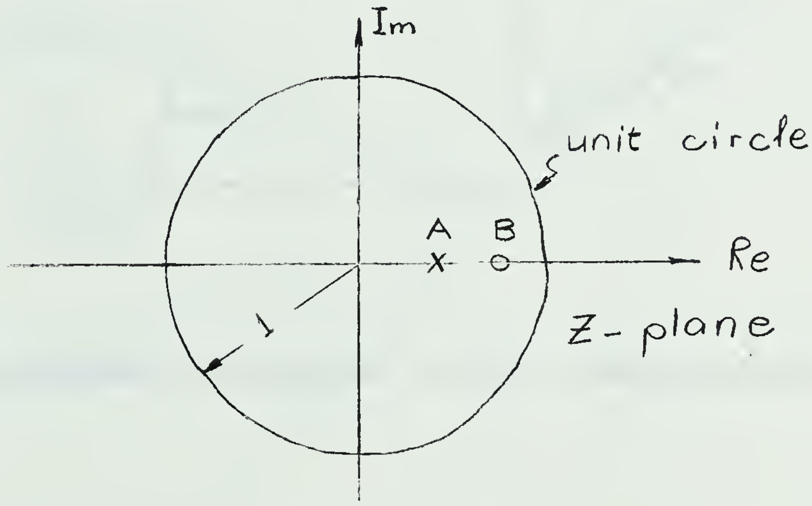


Figure 16. Pole-zero configuration for the lead-network.

Hence the transfer characteristic is

$$D(Z) = \frac{Z-B}{Z-A} .$$

Such a filter has a minimum amplitude of $\frac{1-A}{1-B}$ at $\omega T = 0$, and a maximum value of $\frac{1+A}{1+B}$ at $\omega T = \pi$. The phase reaches a maximum value of lead of $\phi_{max} = \sin^{-1} \frac{B-A}{1-AB}$ which occurs at the frequency of $\omega_m T = \cos^{-1} \frac{A+B}{1+AB}$. These relationships are plotted in figure 17.

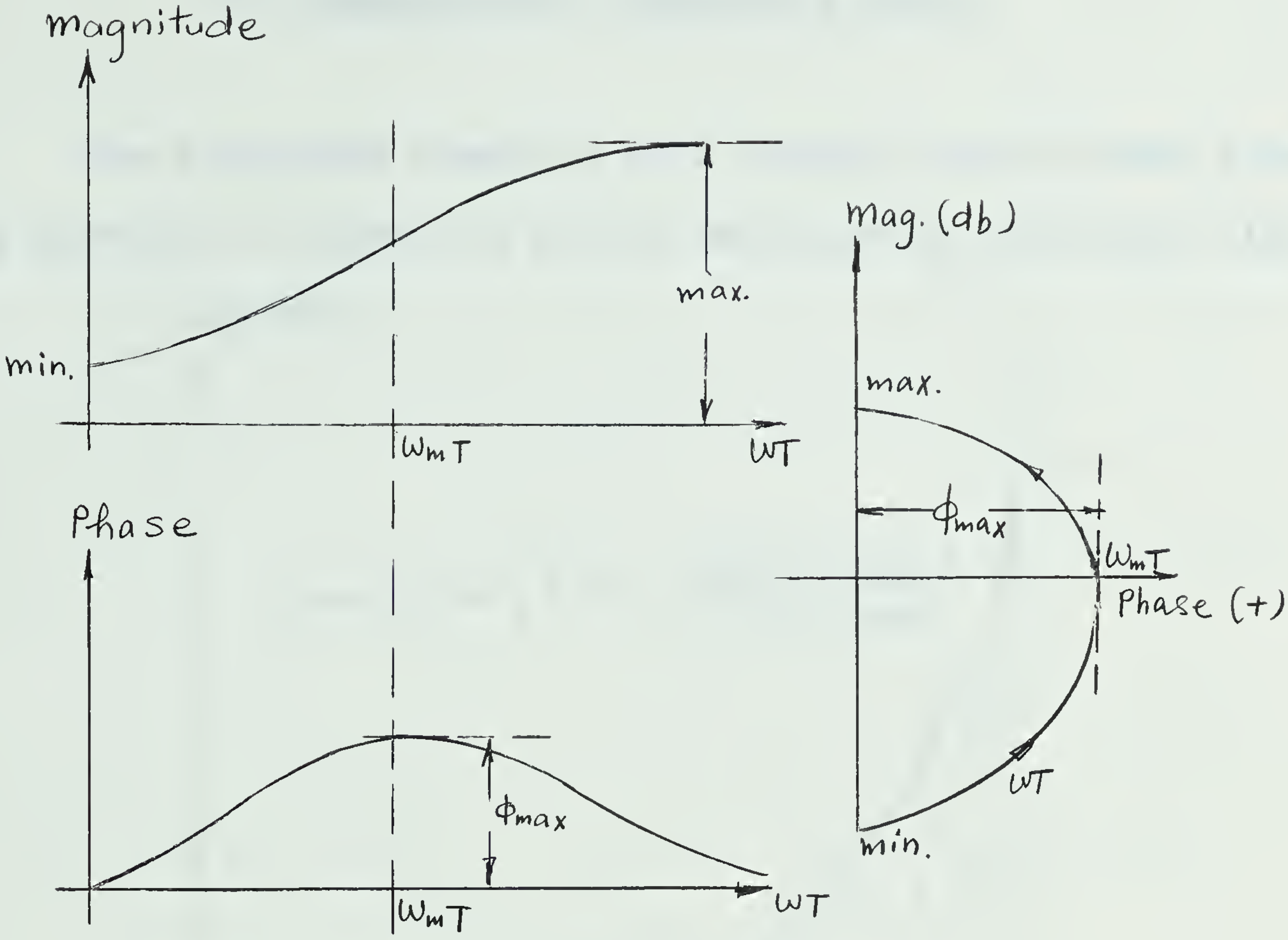


Figure 17. Frequency response of the lead-network.

6. Compensation (Tsypkin's Case)

The frequency response of a linear second-order plant is plotted in figure 18 and is depicted by the solid line.

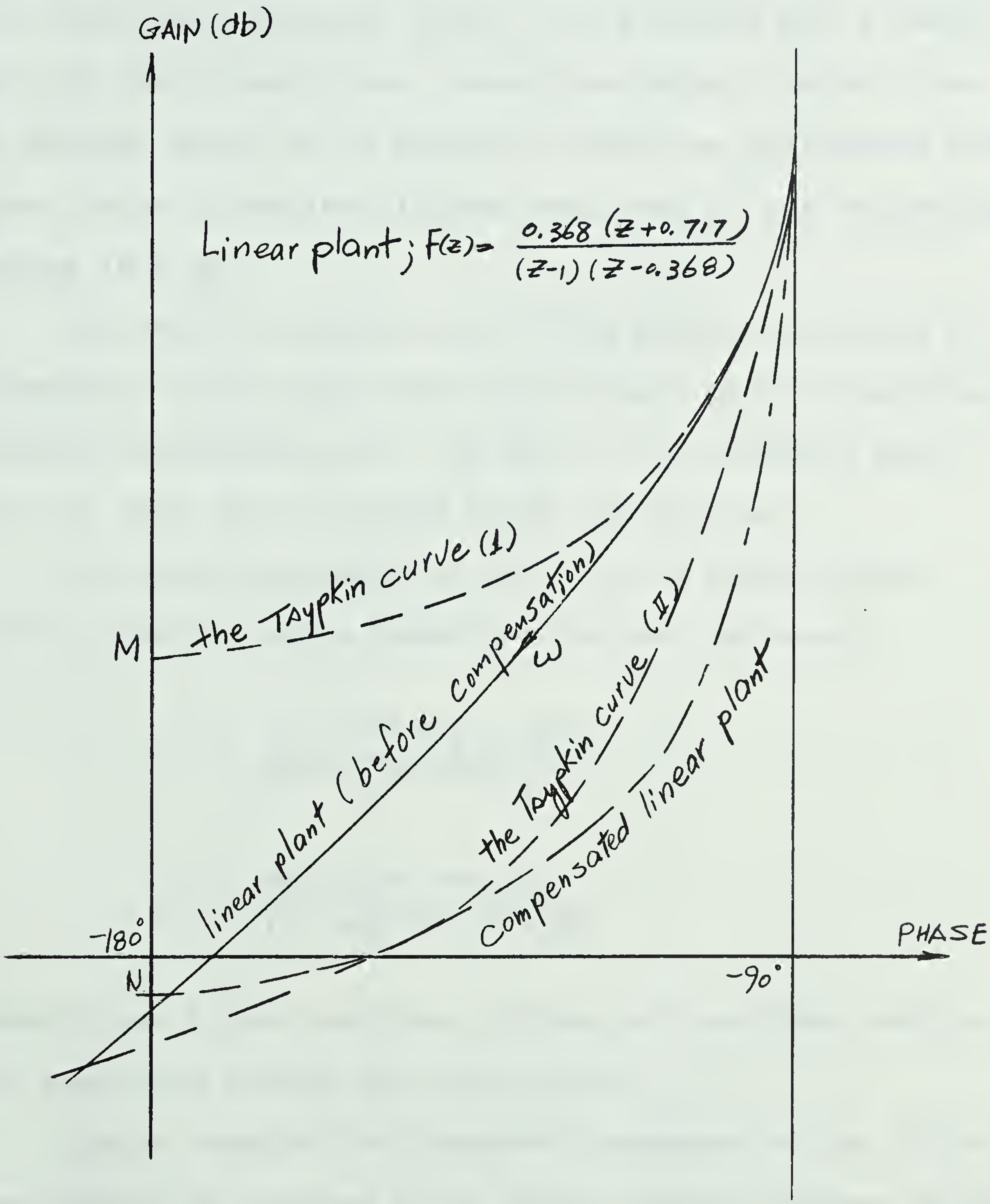


Figure 18. Compensation of system with Tsypkin's criterion.

By incorporating the properly chosen lead-network, the original gain-phase plot is shifted down in amplitude for relatively constant phase. This allows for a larger gain of the linear plant, since the Tsypkin curve 1 can be shifted until it is tangent to the new gain-phase plot; thus giving a smaller minimum real part of the compensated system ($N < M$).

All work is carried out in the Z-domain and the information is obtained from a gain-phase plot without any further transformation. The use of the ordinary Bode plot in this case is found to be impossible.

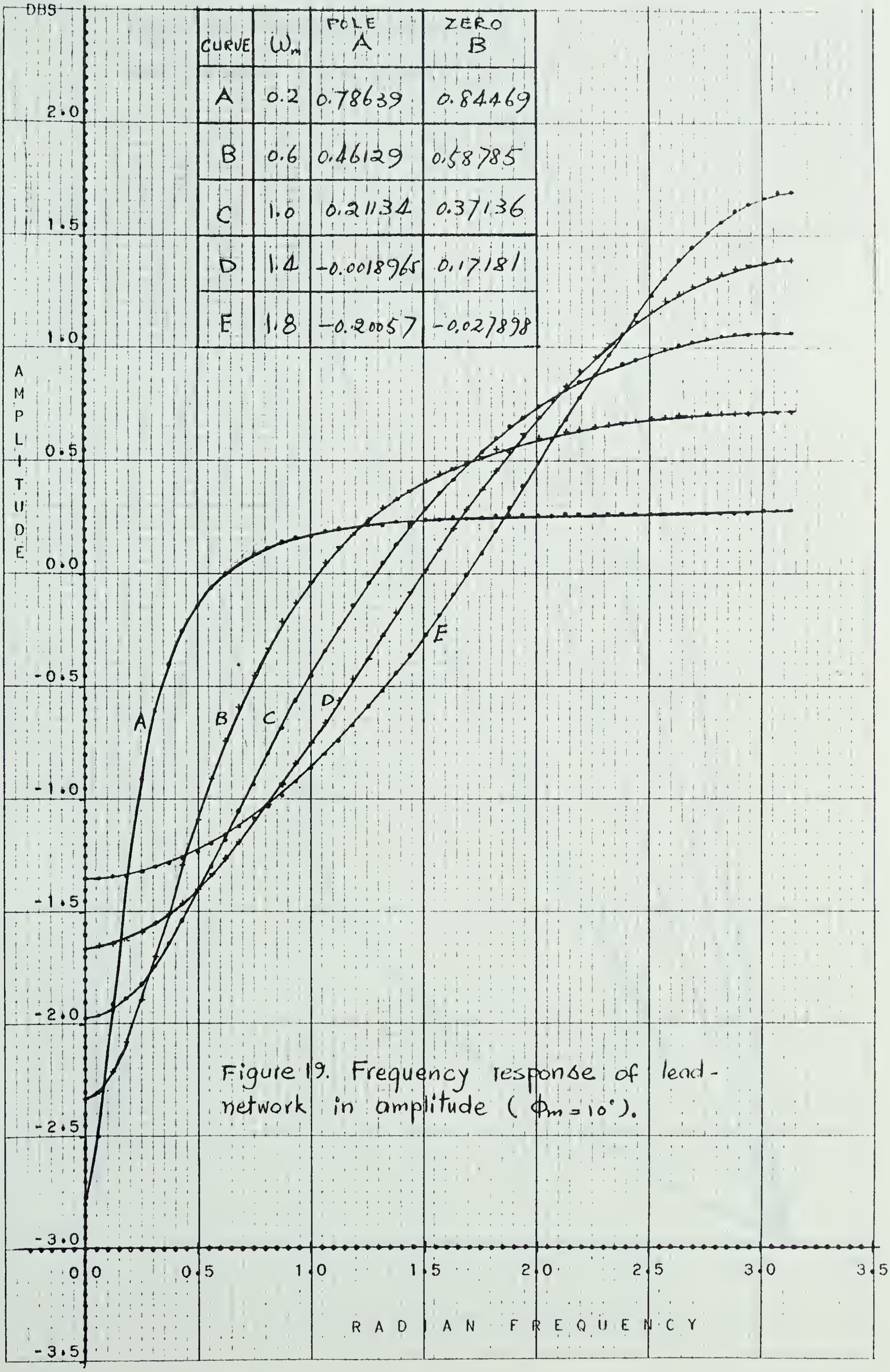
For the convenience of design of a proper lead-filter, the following relationships are necessary:

$$B = \frac{1 + \sin(\phi_m - \omega_m T)}{\sin \phi_m + \cos \omega_m T}$$

$$A = \frac{\sin \phi_m - \cos \omega_m T}{-1 + \sin(\phi_m - \omega_m T)}$$

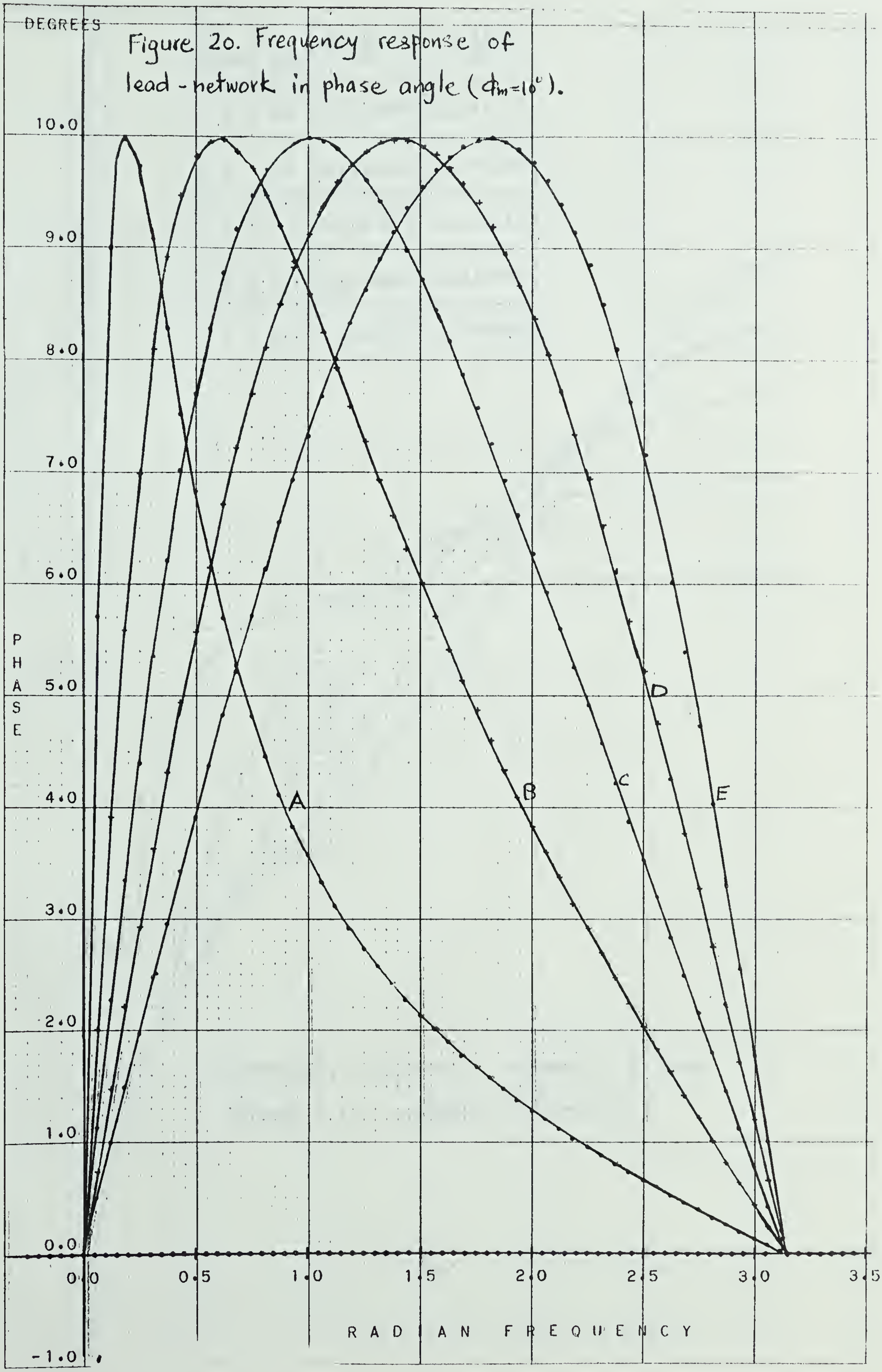
where B and A are locations of the pole and the zero on the real axis inside the unit circle.

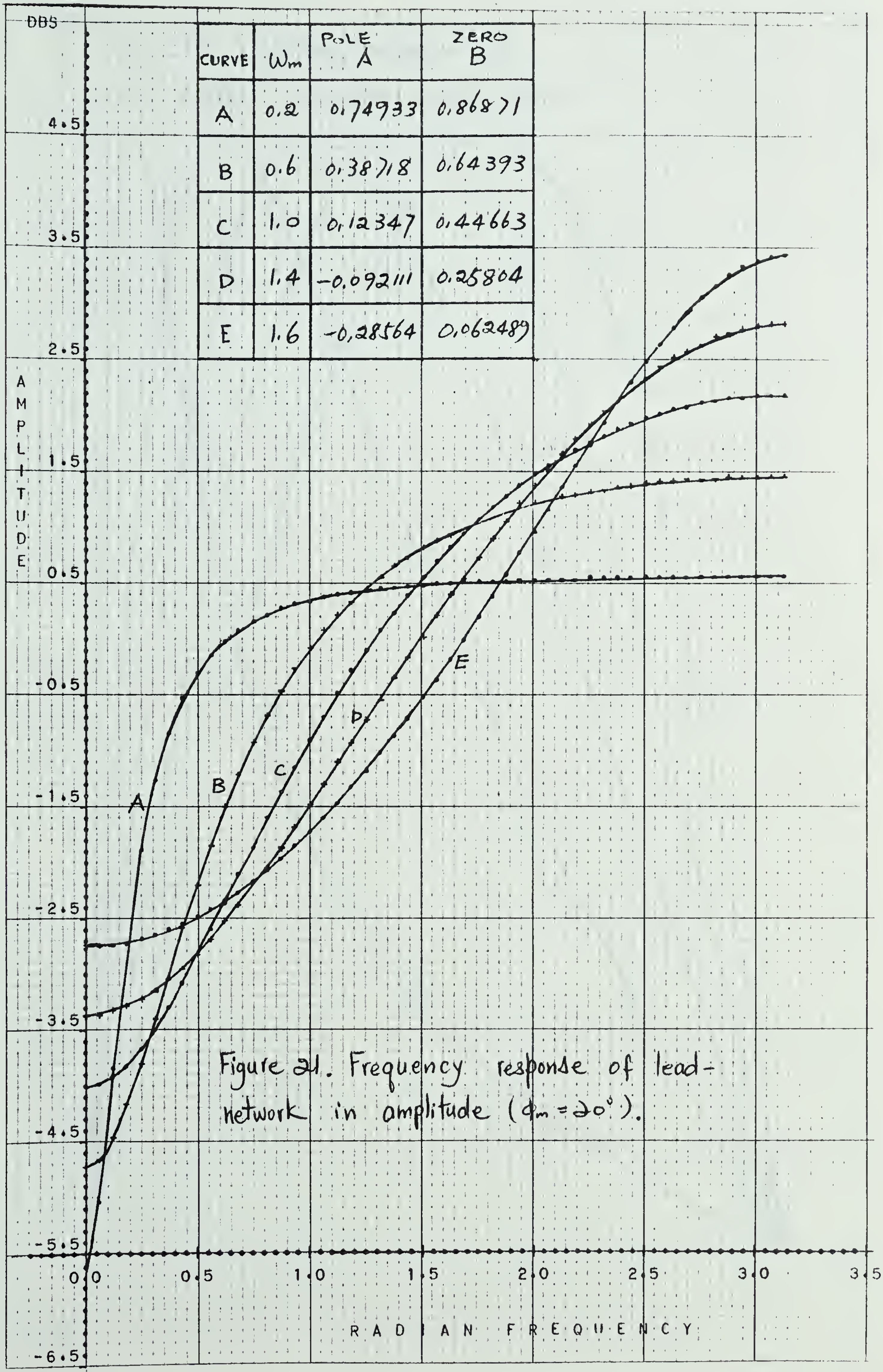
Graphs showing the frequency response of the filter are plotted in figures 19 to 26 for various values of $\omega_m T$ and ϕ_m which determine the locations of pole and zero of the filters to be used as compensators.



CURVE	ω_m	POLE A	ZERO B
A	0.2	0.78639	0.84469
B	0.6	0.46129	0.58785
C	1.0	0.21134	0.37136
D	1.4	-0.0018965	0.17181
E	1.8	-0.20057	-0.027898

Figure 19. Frequency response of lead-network in amplitude ($\phi_m = 10°$).

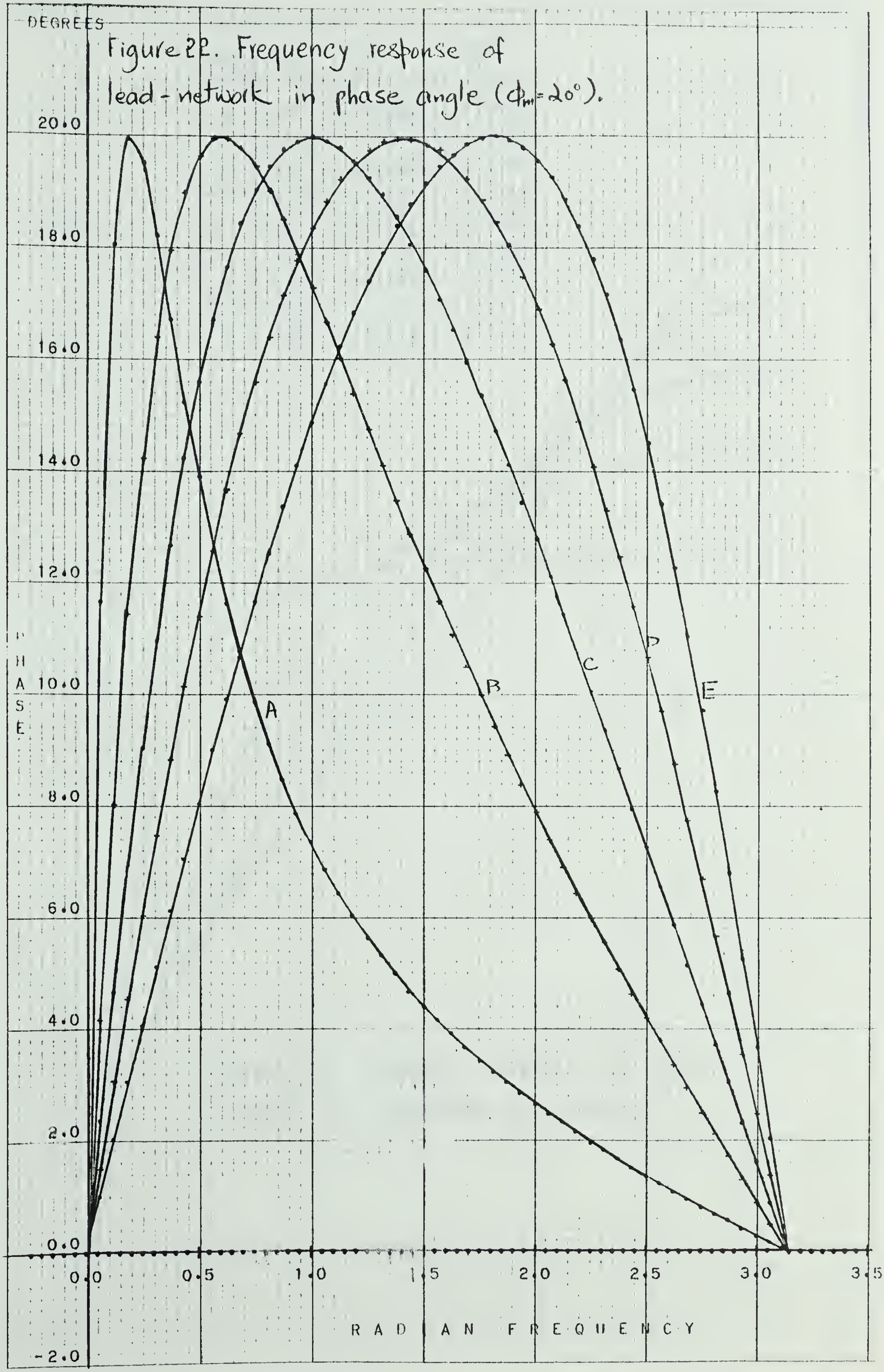

DEGREES
Figure 20. Frequency response of lead-network in phase angle (ϕ_m=10°).
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
PHASE
A
B
C
D
E
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
RADIAN FREQUENCY

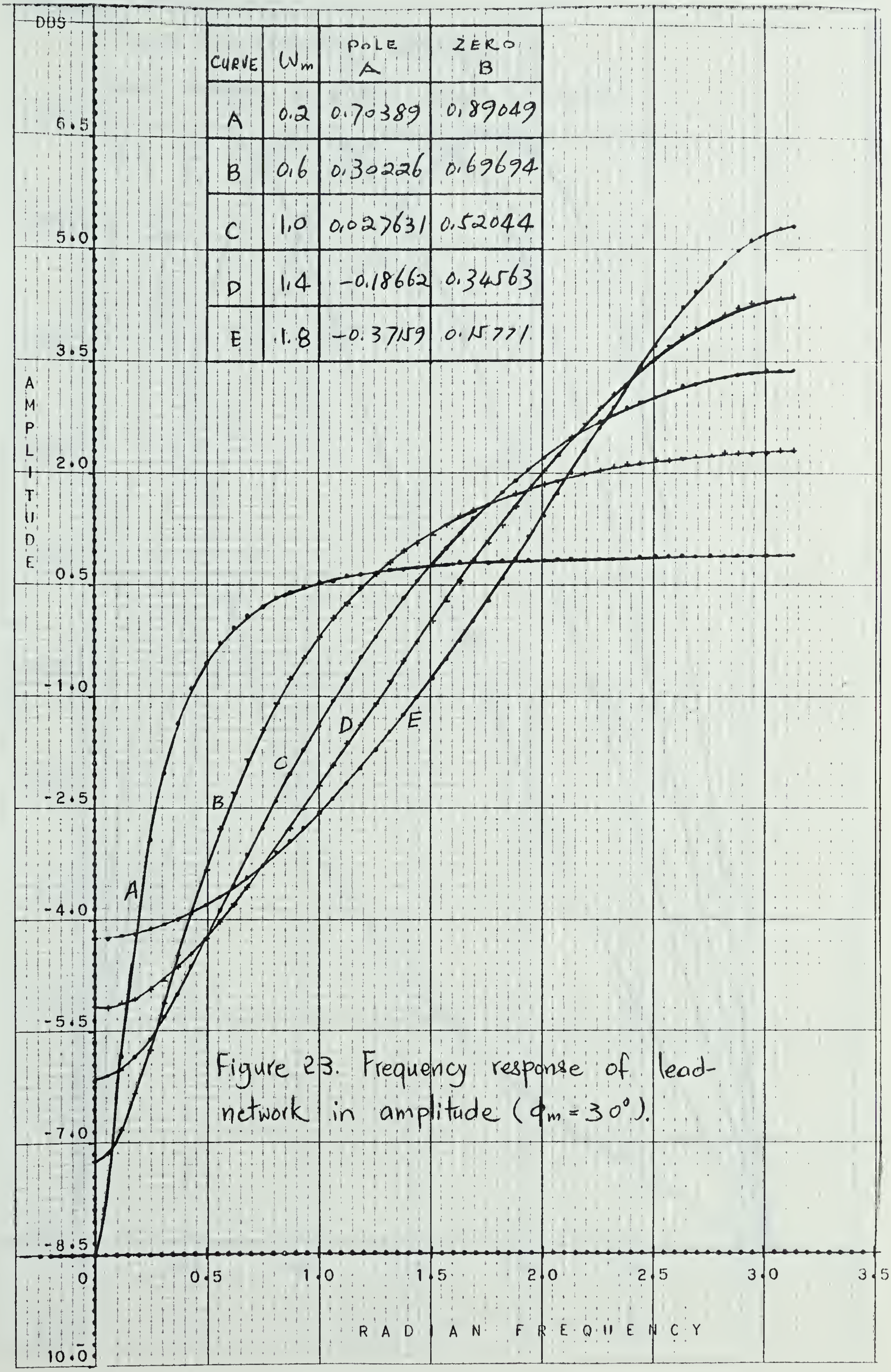


CURVE	ω_m	POLE A	ZERO B
A	0.2	0.74933	0.86871
B	0.6	0.38718	0.64393
C	1.0	0.12347	0.44663
D	1.4	-0.092111	0.25804
E	1.6	-0.28564	0.062489

Figure 21. Frequency response of lead-network in amplitude ($\phi_m = 20°$).

Figure 22. Frequency response of lead-network in phase angle ($\phi_m = 20°$).





CURVE	ω_m	POLE A	ZERO B
A	0.2	0.70389	0.89049
B	0.6	0.30226	0.69694
C	1.0	0.027631	0.52044
D	1.4	-0.18662	0.34563
E	1.8	-0.37159	0.15771

Figure 23. Frequency response of lead-network in amplitude ($\phi_m = 30°$).

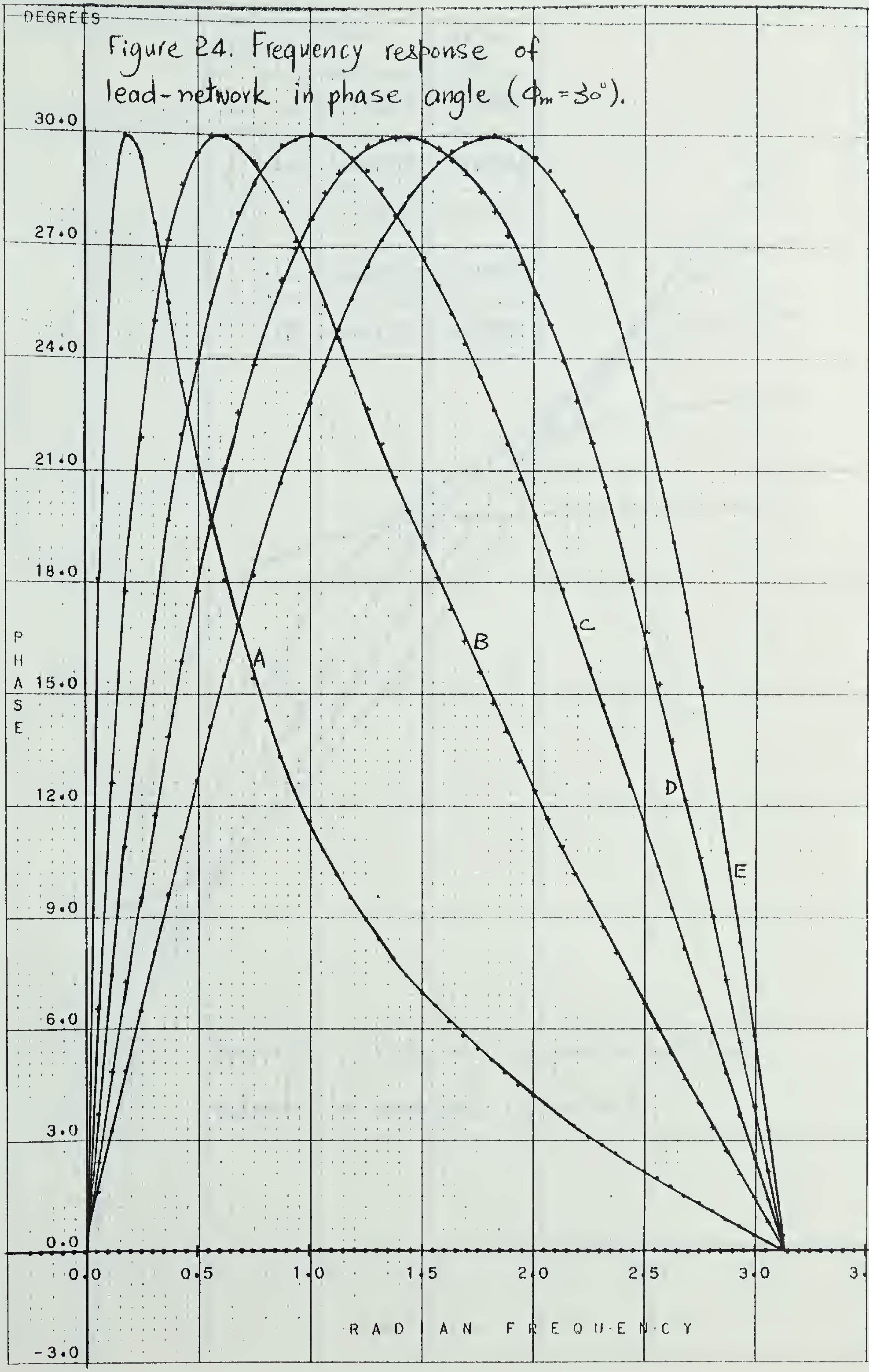
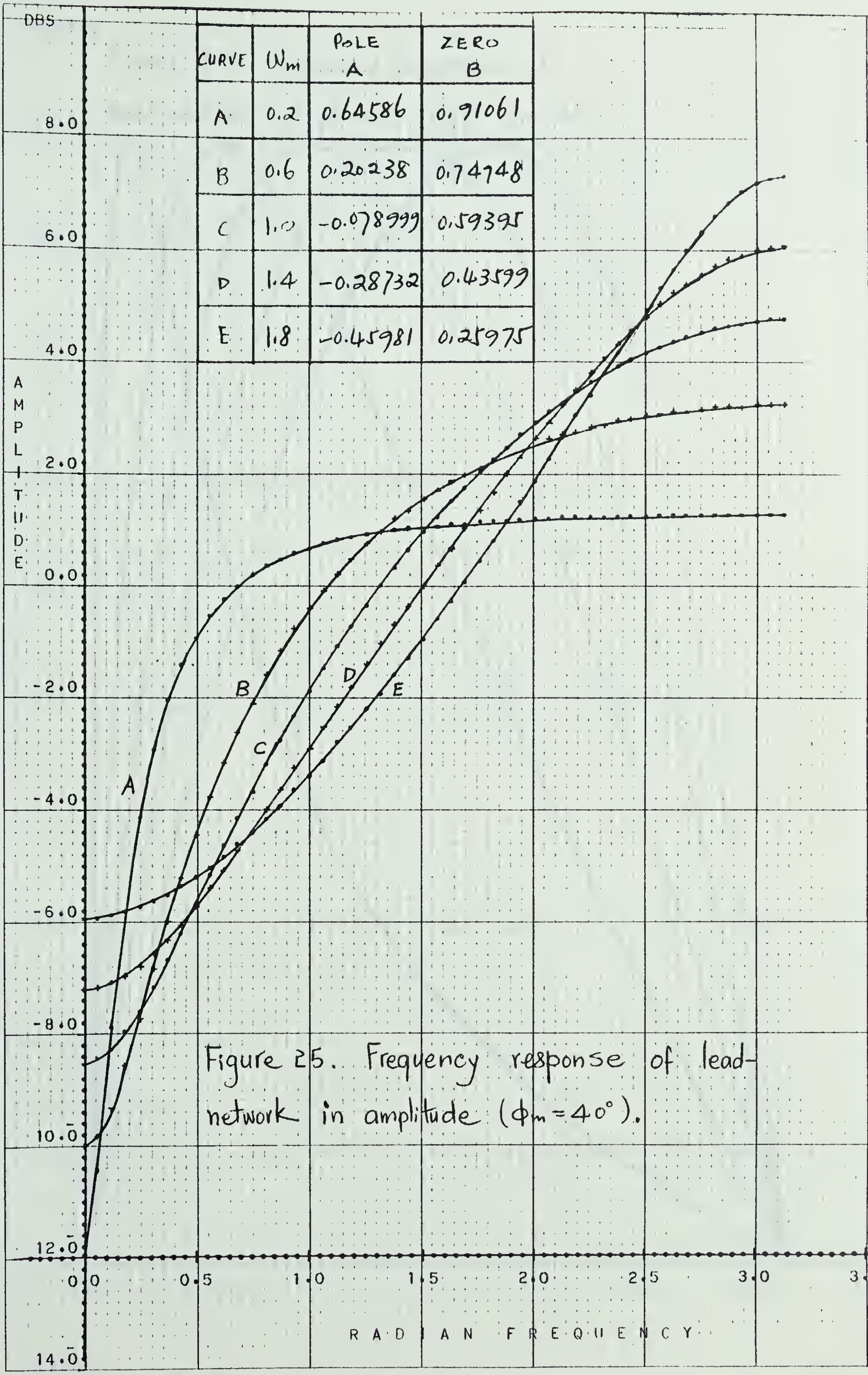
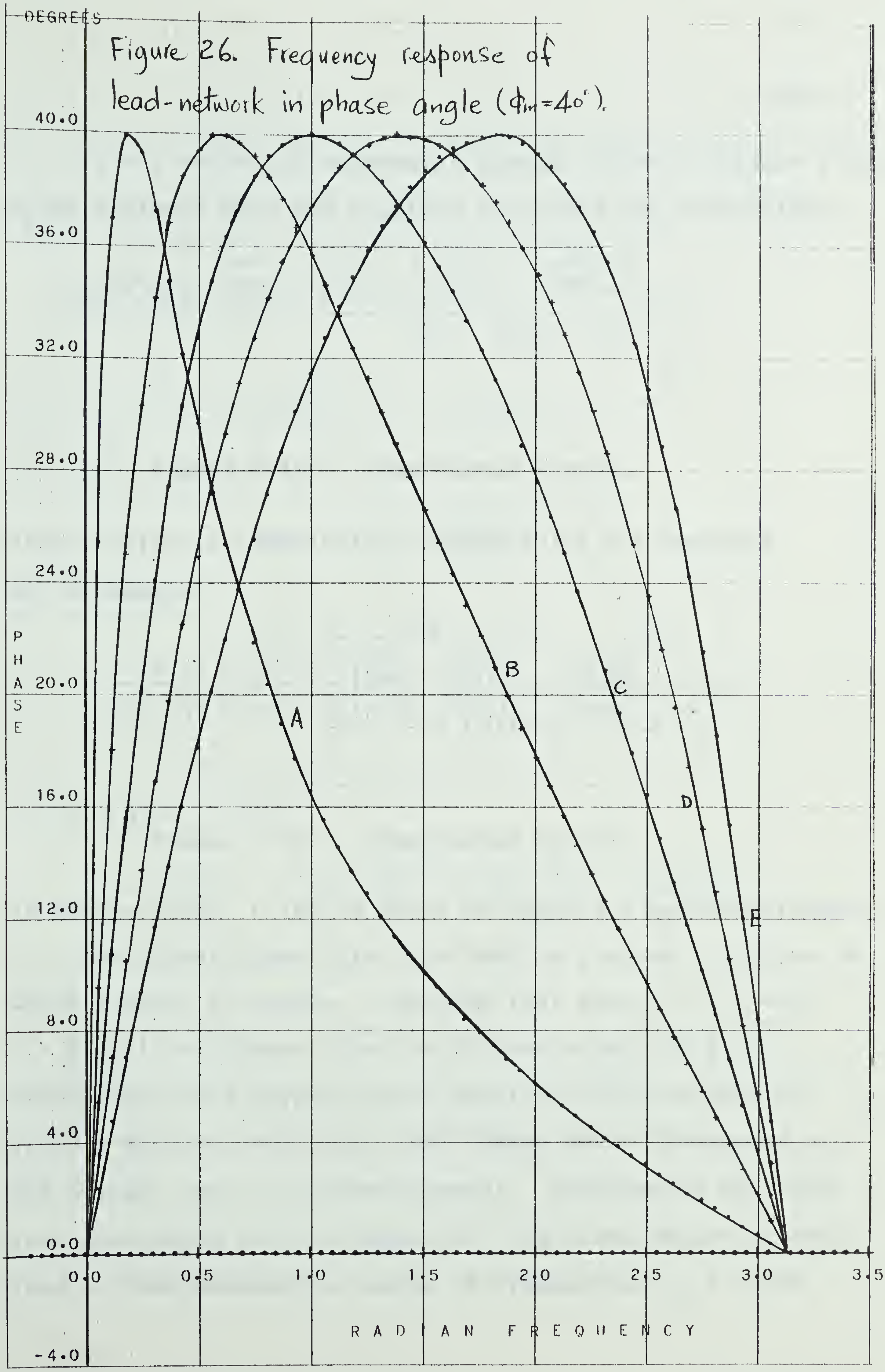


Figure 24. Frequency response of lead-network in phase angle ($\phi_m = 30°$).



CURVE	ω_m	POLE A	ZERO B
A	0.2	0.64586	0.91061
B	0.6	0.20238	0.74748
C	1.0	-0.078999	0.59395
D	1.4	-0.28732	0.43599
E	1.8	-0.45981	0.25975

Figure 25. Frequency response of lead-network in amplitude ($\phi_m = 40°$).


DEGREES
Figure 26. Frequency response of lead-network in phase angle (ϕ_m=40°).
40.0
36.0
32.0
28.0
24.0
20.0
16.0
12.0
8.0
4.0
0.0
-4.0
PHASE
A
B
C
D
E
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
RADIAN FREQUENCY

In the following autonomous system, shown in figure 27(a), $F_c(Z)$ includes both the original plant and the lead-filter.

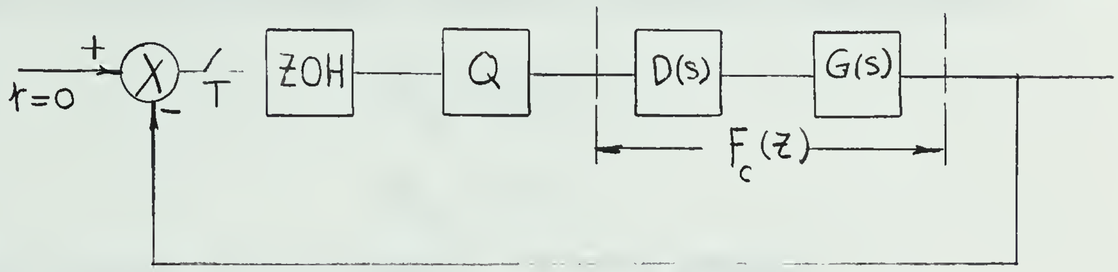


Figure 27(a). Compensated system.

Figure 27(a) is identical to figure 27(b) for analysis and synthesis.

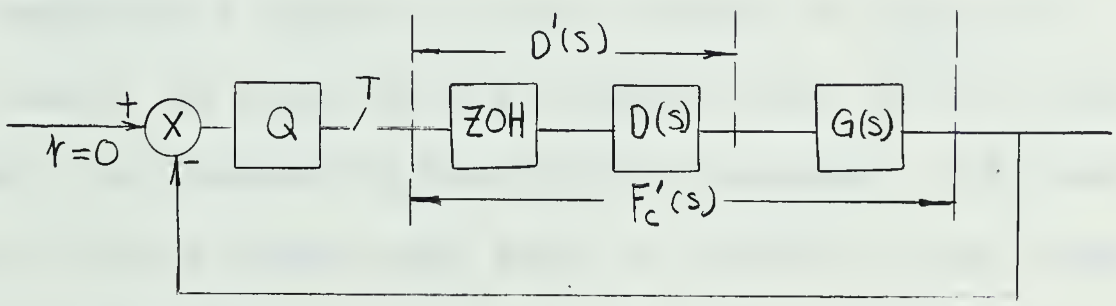


Figure 27(b). Compensated system.

In figure 27(b), $D'(s)$ is shown to contain a zero-order-hold.

The linear plant $\left[G(s) \text{ and ZOH}\right]$ is plotted in figure 28 which yields, as before, a minimum real part of -1.5 with $K = \frac{1}{3}$. If the linear plant is to have a gain of $\frac{1}{2}$, it implies that the Tsypkin curve should be shifted down to yield a gain of zero db at -180° phase shift (indicated by the dotted line in the same figure). Considering the original gain-phase plot of figure 28, the curve should be modified so that between the range of frequencies $\omega_1 = 0.664$

and $\omega_2 = 0$, the Tsypkin curve should be above the gain-phase plot for all phase angles in order to maintain absolute stability. Judging from figures 19 to 26, by trial-and-error, the following optimal values are picked:

$$\phi_m = 10°$$

$$\omega_m = 0.2$$

which yield A = 0.78639, B = 0.84469.

Therefore, the lead-filter is

$$D(Z) = \frac{Z - 0.84469}{Z - 0.78639}$$

The compensated system is also plotted in figure 28. This result is close to the required gain of the plant. In fact, as checked by the digital computer, the exact value of the minimum real part is -0.9755. The compensated system is now

$$F_c(Z) = \frac{0.368K(Z + 0.717)\ (Z - 0.84469)}{(Z-1)\ (Z - 0.368)\ (Z - 0.78639)}$$

where $K = \frac{1}{2}$.

At this point, it may be appropriate to clarify the relative locations of the various components in the system. The original system without compensation is shown in figure 29.

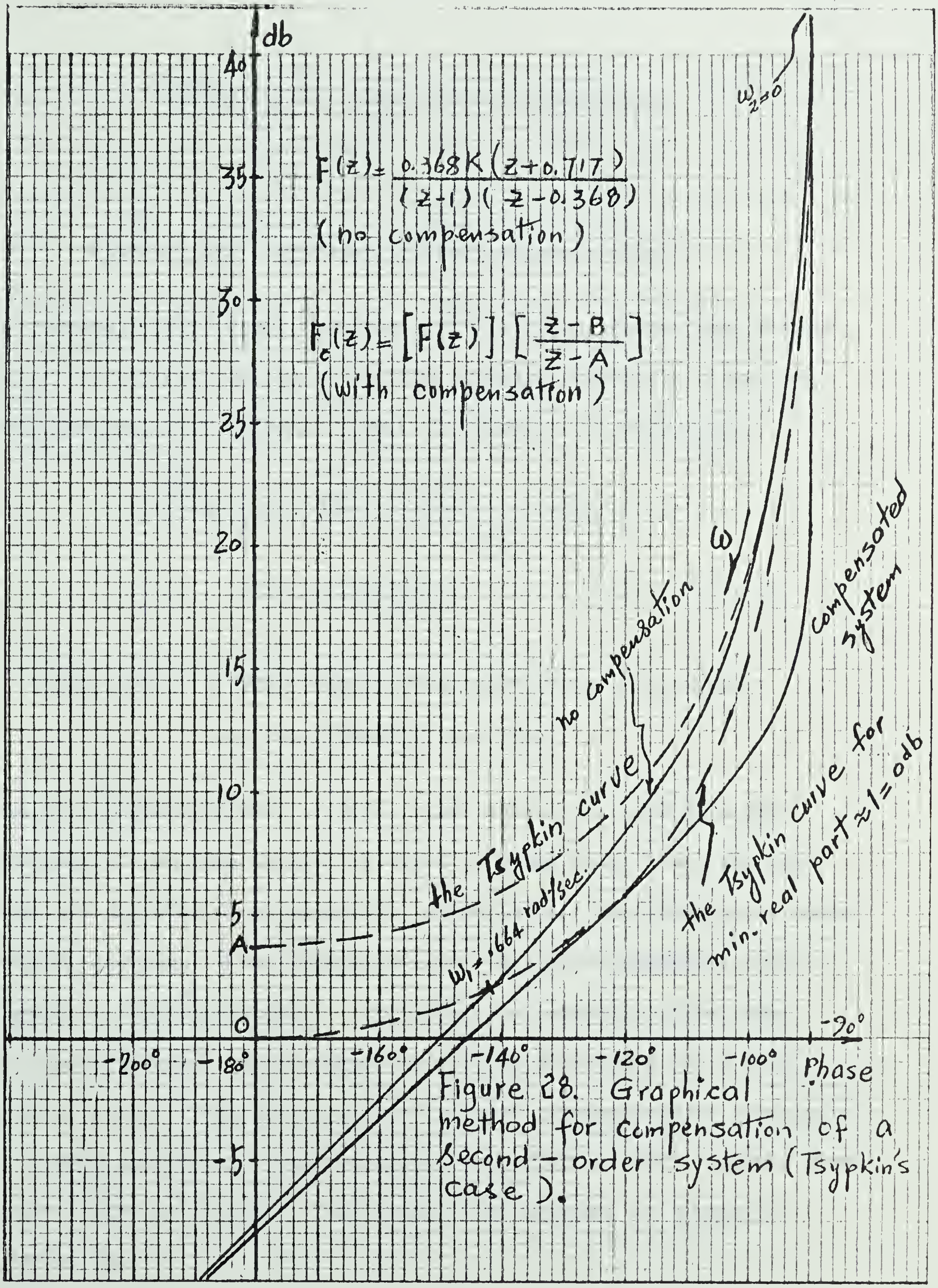


Figure 28. Graphical method for compensation of a second-order system (Tsypkin's case).

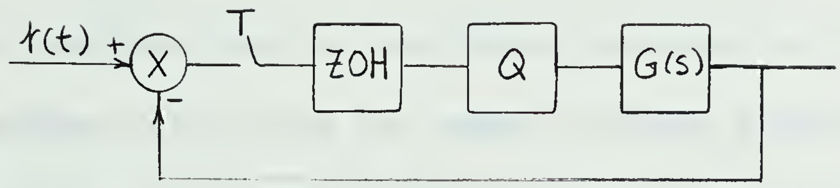


Figure 29. System without compensation.

To convert the block diagram into a form where the stability criterion can be directly applied without changing the system, the scheme shown in figure 30 resulted.

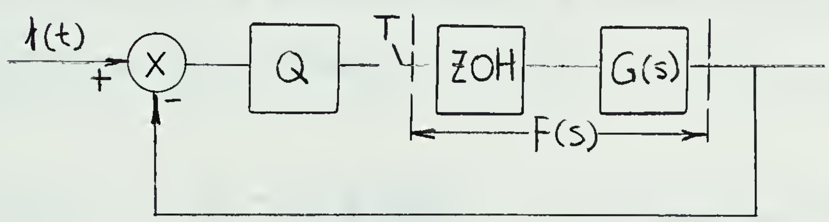


Figure 30. A form where the stability criterion can be directly applied.

The new linear plant is one that contains both the ZOH and G(s). On the basis of this linear part the critical gain is derived for absolute stability. No change is made on the system. For compensation, a lead-network is designed in the Z-domain. In order to maintain the form of the original error-sampled system followed by a ZOH, and in order that the system without compensation consists of a ZOH and G(s), the compensating filter D(Z) should contain a ZOH in front. The open-loop system is given in figure 31.

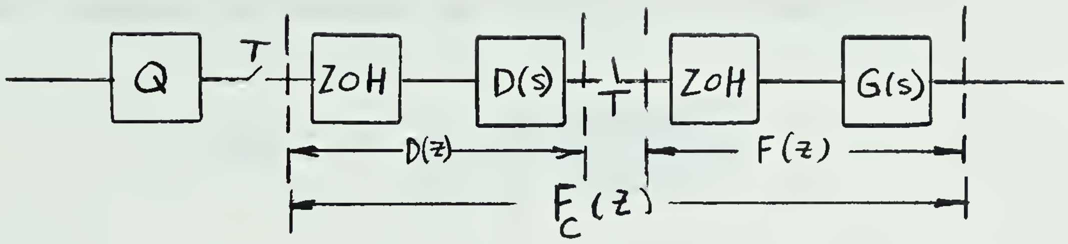


Figure 31. Open-loop system with compensation.

In the system shown in figure 31, the arrangement of the block diagram is made for convenience in theoretical work and the actual open-loop system takes on the form shown in figure 32.

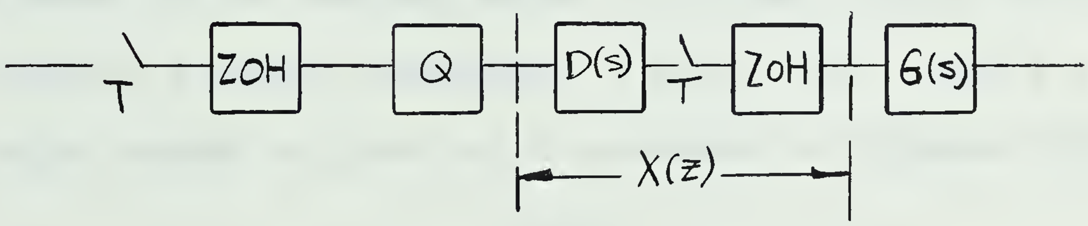


Figure 32. Actual open-loop system after compensation.

X(Z) is the effective compensation. Considering the scheme in figure 31 where $D(Z) = \frac{Z-B}{Z-A}$ (B > A), it is found to have the arrangement shown in figure 33. Note that $D(Z) \neq X(Z)$.

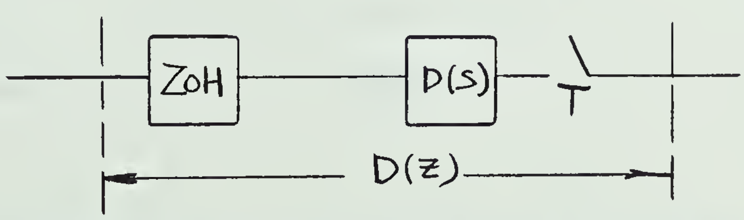


Figure 33. Block diagram of compensator.

D(s) can be synthesized by ordinary methods of realization, since the number of poles is equal to the number of zeros, and the pole and zero are both simple, real and inside the unit circle in the Z-plane. The zero is always positive as is the pole in most cases. When the pole is positive, it is possible to realize D(s) by a series pulsed-data network or a digital computer. Otherwise, if the pole is negative, D(s) may be realized on a digital computer.

Considering the example where

$$D(Z) = \frac{Z - 0.84469}{Z - 0.78639}\,, \quad \text{it can be realized}$$

for the series case shown in figure 34.

Figure 34. Realization by a series pulsed-data network.

For the feedback pulsed-data network, it is always possible for a realizable zero, however, in the actual arrangement of the system, the ZOH cannot be separated from D(s). Therefore, this method is disregarded.

For the time-delay digital programming method, in order to separate a ZOH from D(Z); it can be shown that

$$D(Z) = \frac{Z-B}{Z-A} = D'(Z) \quad \mathcal{Z}\left[\frac{1 - e^{-sT}}{s}\right] = D'(Z) \; .$$

This scheme is shown in figure 35 where

$$D(Z) = \frac{1 - 0.84469Z^{-1}}{1 - 0.78639Z^{-1}}$$

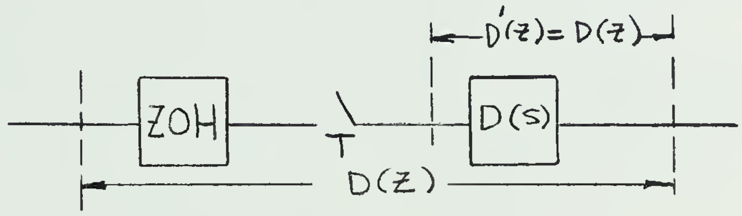


Figure 35. Realization by a digital computer.

Figure 35 is redrawn below in figure 36(b) showing the digital program inserted. Figure 36(a) shows a unit time-delay which is part of the scheme in figure 36(b).

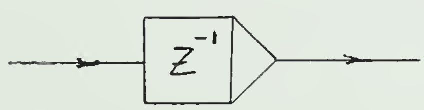


Figure 36(a). A unit time-delay of T seconds.

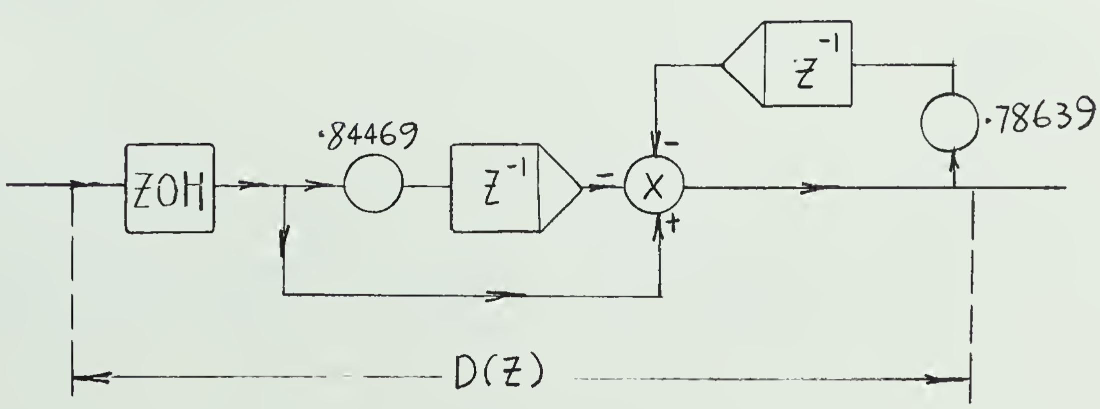


Figure 36(b). Realization by a digital computer.

Addition of a ZOH and a sampler in front of the digital computer does not change the input to G(s), since the output of the digital computer must also be sampled and held.

7 Stability and Compensation (Jury and Lee's Case)

A modified criterion based on Tsypkin's result has been derived by Jury and Lee[4]. They have found a sufficient condition of absolute stability for this kind of system which is much less conservative.

If, for the transfer curve of the memoryless nonlinearity shown in figure 37,

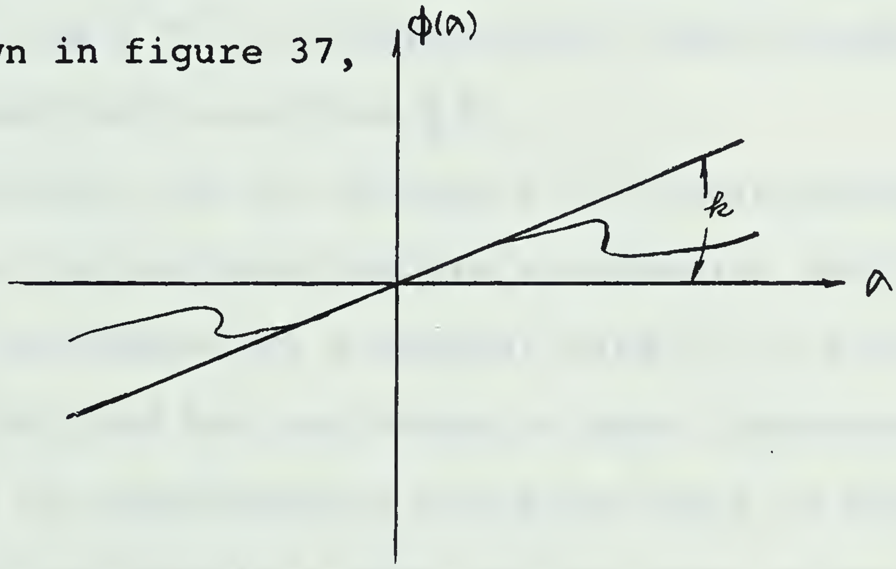


Figure 37. A nonlinearity.

the nonlinear gain function is assumed continuous, and satisfies the conditions;

$$\phi(0) = 0, \quad k \geqslant \frac{\phi(\sigma)}{\sigma} > 0 \quad \text{for} \quad \sigma \neq 0, \quad \left|\frac{d\phi(\sigma)}{d\sigma}\right| < k' ,$$

or $k \leqslant k'$ where k' is the local slope of the nonlinearity*; the linear part must be stable, and $\lim_{s \to \infty} G(s) = 0$.

---

* in the case where the nonlinear function is piece-wise-linear, the condition for the local slope is satisfied if the slope of each linear segment is less than or equal to k .

The theorem states that; such a nonlinear sampled-data system is absolutely stable if a non-negative q exists such that the following condition,

$$\operatorname{Re} F(Z)\left[1 + q(Z-1)\right] + \frac{1}{k} - \frac{k'|q|}{2}\left|(Z-1)\ F(Z)\right|^2 \geq 0$$

is satisfied on the unit circle $Z = e^{j\omega T}$ ... (α). Again the same assumption as before is made that the condition $k > \frac{\phi(\sigma)}{\sigma} > 0$ for $\sigma \neq 0$ is relaxed for finite steady-state error with magnitude less than $\frac{1}{2}\ell$.

It is obvious that by letting q = 0, this theorem is reduced to the previous Tsypkin's criterion which can therefore be considered as a special case of this one. Examples by Jury and Lee are shown in some instances to yield results of approximately twice the gain as compared to those found by Tsypkin's case for the same system.

In a later paper by the same authors(5), the above condition has been extended to some specific subclasses of the basic system. One of the extended cases which is suitable for application to systems containing a quantizer is a modification of condition (α);

$$\operatorname{Re} F(Z)\left[1 + q\,\frac{Z-1}{Z}\right] + \frac{1}{k} - \frac{1}{2}k'q\left|(Z-1)\ F(Z)\right|^2 \geq 0$$

for some non-negative q, where k' is the lower bound on the derivative of $\phi(\sigma)$ and its upper bound is infinite.

The local slope of the quantizing characteristic

curve has a lower bound of zero and an upper bound of infinite value, as required; the quantity k' is thus set to be zero in the above condition. For this quantizing case, the condition becomes;

$$\text{Re } F(Z) \left[1 + q \frac{Z-1}{Z}\right] + \frac{1}{k} \geqslant 0$$

for all points on $|Z| = 1$ and some non-negative q.

The discussion of absolute stability and compensation that follows is based on the last condition.

When the value of k is fixed, say two in our case, a minimum value must be found for $\text{Re } F(Z) \left[1 + q \frac{Z-1}{Z}\right]$ as Z takes on values along $|Z| = 1$ and the non-negative q also changes. An attempt to obtain analytically the optimal value for q is quite tedious and lengthy as the linear part, F(Z), increases its complexity. A digital program to find the optimal q is given in appendix (2) for a fairly general form of F(Z). In this case using the digital computer, the optimal q has been found to be q = 0.870 which yields; $\min \text{Re } F(Z) \left[1 + 0.87 \frac{Z-1}{Z}\right] = -0.7718$. Again, the same graphical method as developed in section five and six is used here for obtaining stability and compensation. Referring to figure 38, with the aid of the Tsypkin curve, the minimum real part is found to be -0.7715. The maximum gain allowable (for k = 2) is 0.648 which almost doubles the previous value of 1/3 in Tsypkin's case (q = 0). If

a gain of $K = 1.16$ is required, it is required that

min Re $F_c(Z) = \frac{1}{(2)(1.16)} = 0.432$. To be safer, the Tsypkin curve is shifted down cutting the 180° axis at approximately -7.5 db = 0.4265 (180°) as shown in figure 38.

In this case, compensation is expected to be more difficult due to the fact that the original gain without compensation is already high. However, by trial-and-error, using the same technique as before, the following optimal values were selected;

$$\phi_m = 30° , \qquad \omega_m = 2.05$$

The compensated curve, also plotted in figure 38, shows that the resultant curve is below the specified Tsypkin curve for all frequencies. The minimum real part obtained on the digital computer is -0.4267.

For the above values of ϕ_m and ω_m chosen, locations of pole and zero are given as; A = -0.4807732, B = 0.025311336 so that:

$$D(Z) = \frac{Z - 0.025311336}{Z - 0.4807732}$$

The procedure for synthesis is no different from the previous one. Since we have a negative pole, the only possible way of realization is by a digital computer as shown below in figure 39.

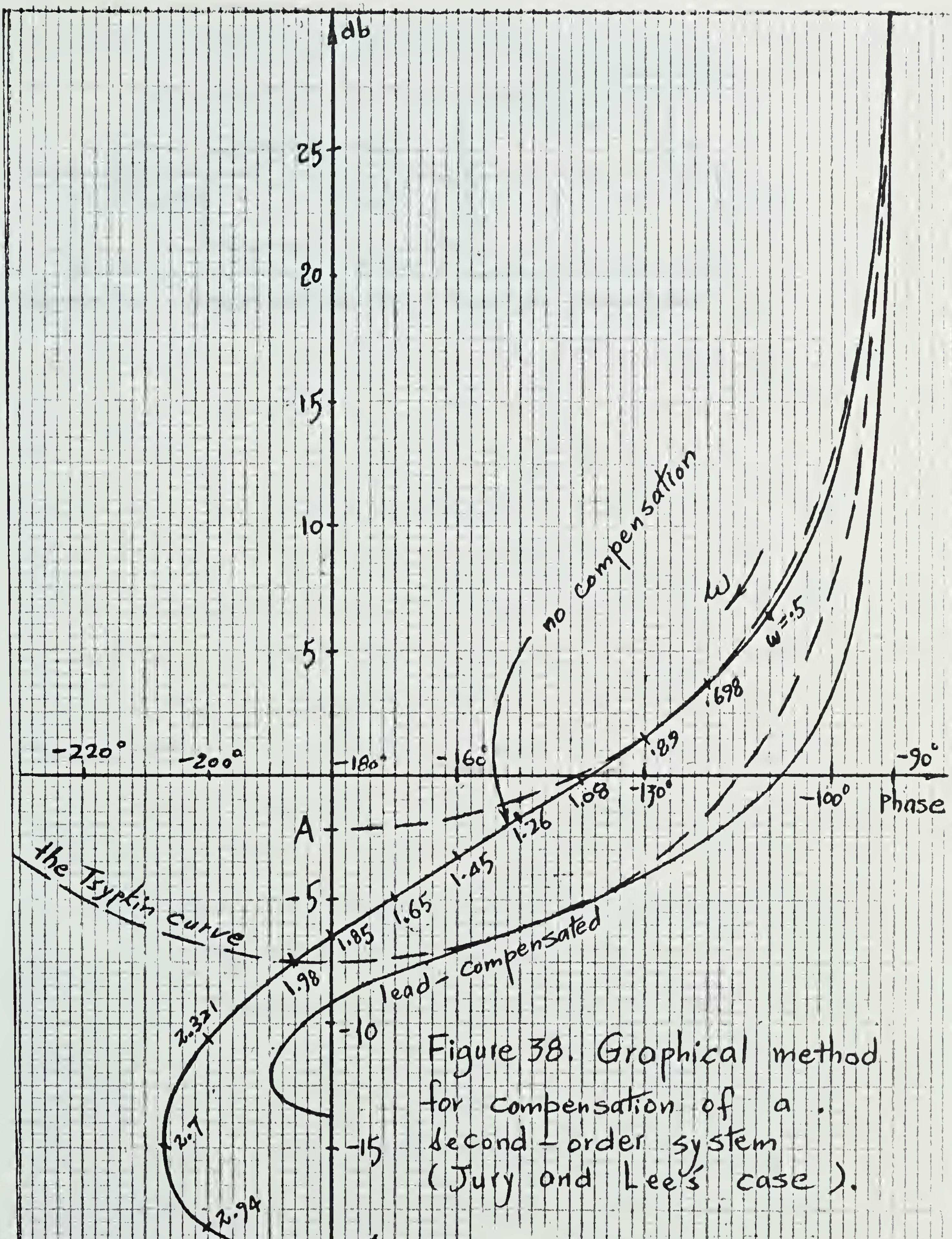


Figure 38. Graphical method for compensation of a second-order system (Jury and Lee's case).

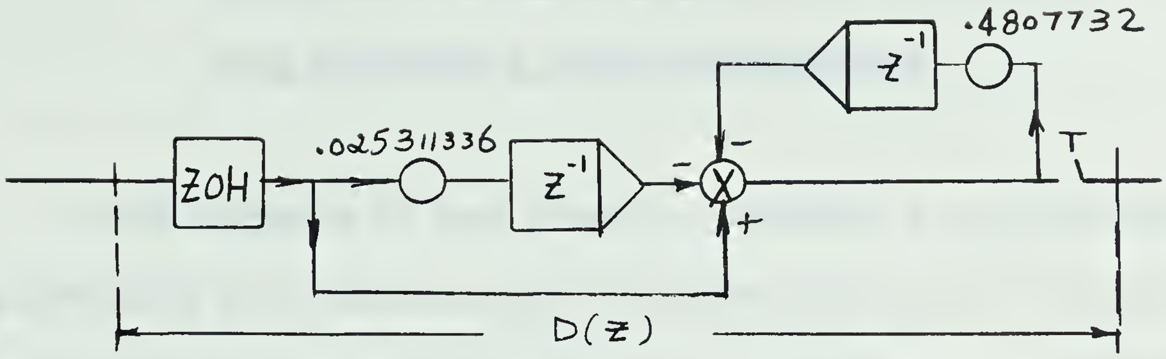


Figure 39. Realization by a digital computer.

8 Conclusion of the Graphical Technique for Stability and Compensation

The objective has been to present a new graphical procedure for determining absolute stability in the large and compensation of Tsypkin's case and the modified case by Jury and Lee. In particular, for the compensation with respect to the gain of the linear plant, this method is quite satisfactory. No attempt was made to obtain it analytically, due to the mathematical difficulty in this kind of system. All necessary information in terms of frequency response is converted into a gain-phase diagram and the whole procedure is carried out in the Z-domain without any further transformation. The results so obtained graphically agree very closely with those worked out on a digital computer. Physical realization of the lead-network so developed is shown to be possible. In Jury and Lee's case, some programming work was performed to obtain the optimal q which yielded the least conservative result. This method is general as far as the shape of the transfer characteristic of the nonlinearity is concerned. However, for finite local slopes, the last term in Jury and Lee's criterion cannot be dropped. In that case, the digital program must be slightly modified to find the optimal q. All other steps remain essentially the same.

Referring to noise in the practical system or inaccuracy of the quantizer constructed, the stability condition has only to be slightly modified to take this into consideration. For example, in our case, for ± 10 m.v. of inaccuracy of the electromechanical relay used the critical gain is modified as follows;

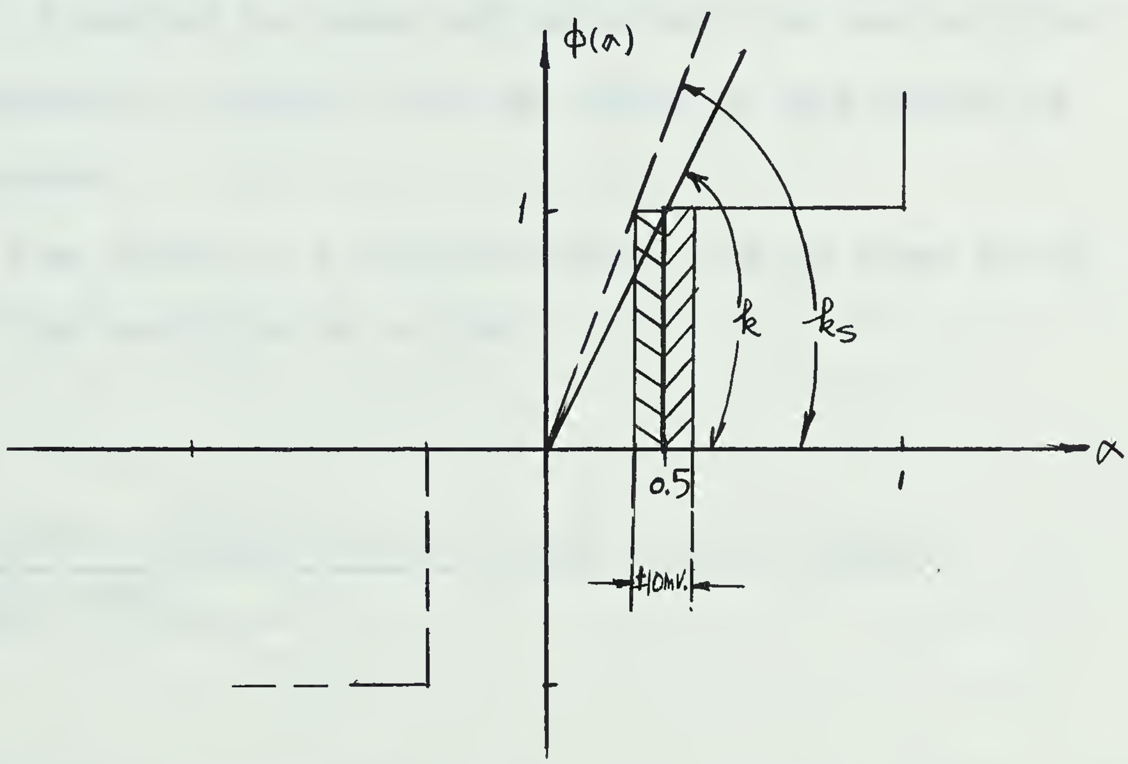


Figure 40. Modification for inaccuracy of quantizer.

the new sector; $k_s = \dfrac{1}{0.5 - 0.01} = 2.04$. The critical gain is accordingly reduced by two percent.

Although, a second-order system with unity feedback has been used to serve the purpose of an illustrating example throughout the discussion, the usefulness of the graphical method is not necessarily limited to such a particular system.

9 Appendix

1 Scheme for starting the system at the sampling instant

In working with the analog computer on sampled-data systems, a method is employed to start the system right at the sampling instant with an error of the order of microseconds.

If the input is a step-function, it is easy to do this by the addition of a ZOH.

The actual input to the system starts only at the instant of sampling. However, for other time-varient inputs such as ramps, or sinusoidal inputs, a single ZOH is not adequate for this purpose. An electronic comparator is used in the new scheme as shown in figure 41.

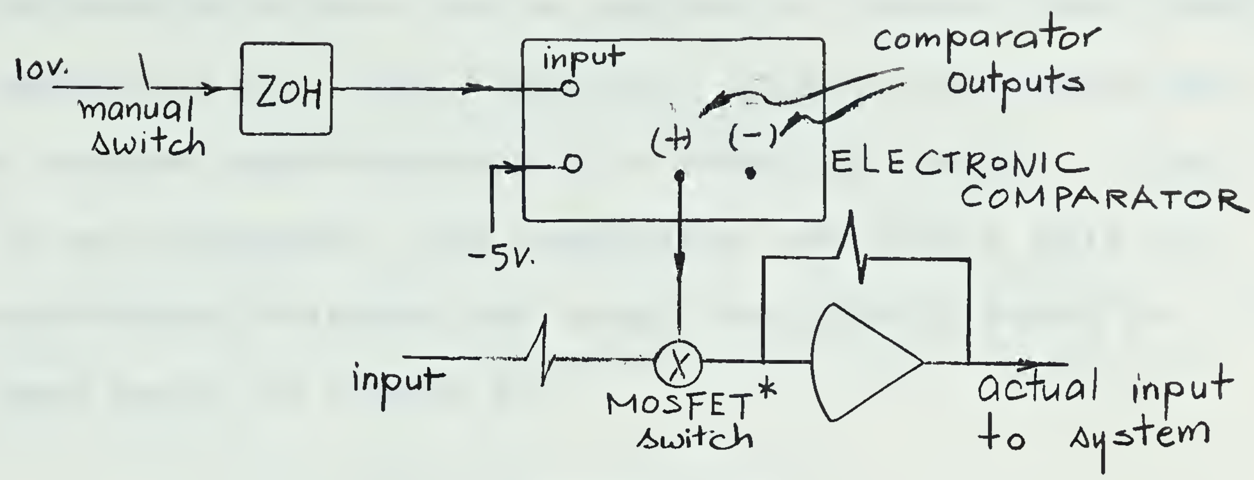


Figure 41. Scheme for starting the system at the sampling instant.

When the sum of inputs to the electronic comparator is positive, the output marked by a "+" sign supplies a positive voltage high enough to close the switch in the amplifier circuit. Before the manual switch is closed, the output of the ZOH is zero; the input to comparator is -5 v.; the "+" sign output is negative; the switch in the amplifier is open and no input to the system is present. When the manual switch is closed, the output of the ZOH is not +10 v. until the sampling instant, at which the net input to the comparator is +5 v. When the switch in the amplifier is closed, this gives whatever input is applied to the system. Once the manual switch is closed, the switch in the amplifier remains closed.

---

* MOSFET: metal-oxide silicon field-effect transistor.

The same principle can be applied to control the circuit generating the input functions, so that everything can be started synchronously at a sampling instant. Also, for an integrator, the comparator can form a pair of synchronous switches for reset and operate modes as shown below in figure 42.

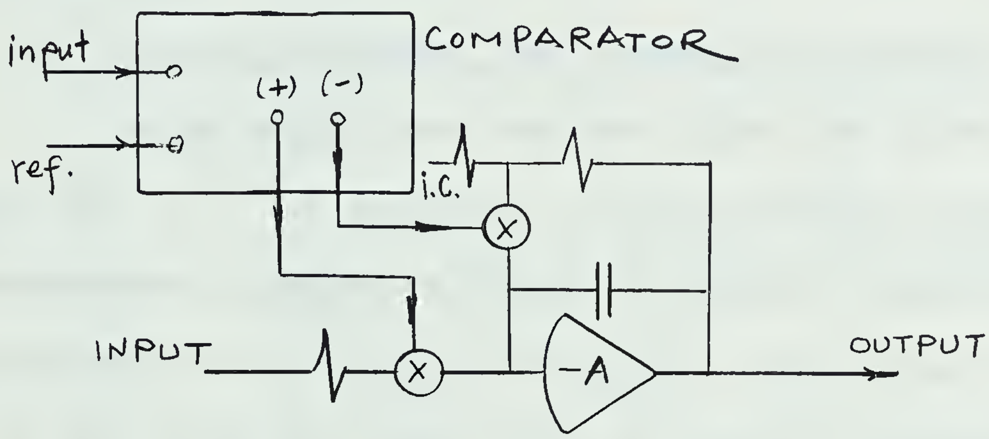


Figure 42. Electronic comparator used to control the reset and operate modes.

Four electronic comparators were made together with eight MOSFET switches and associated protecting circuits. The overall switching time is in the order of microseconds and the dead-zone accuracy is $\pm$ 10 m.v. No attempt was made to obtain greater accuracy or speed, since the amplifier noise on the computer is about 10 m.v. and the sampling time controlled by the pulses is one millisecond.

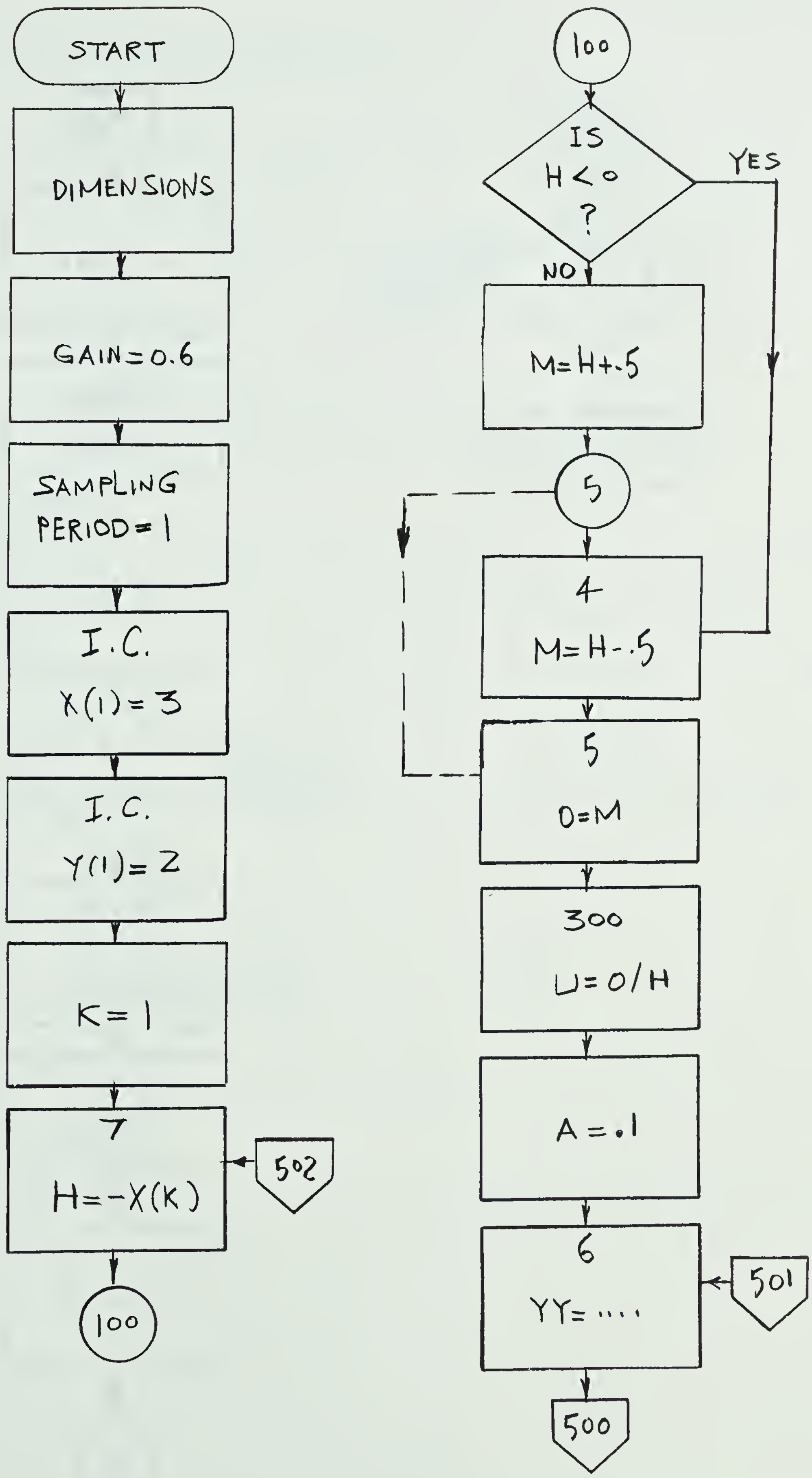
2 Digital programs

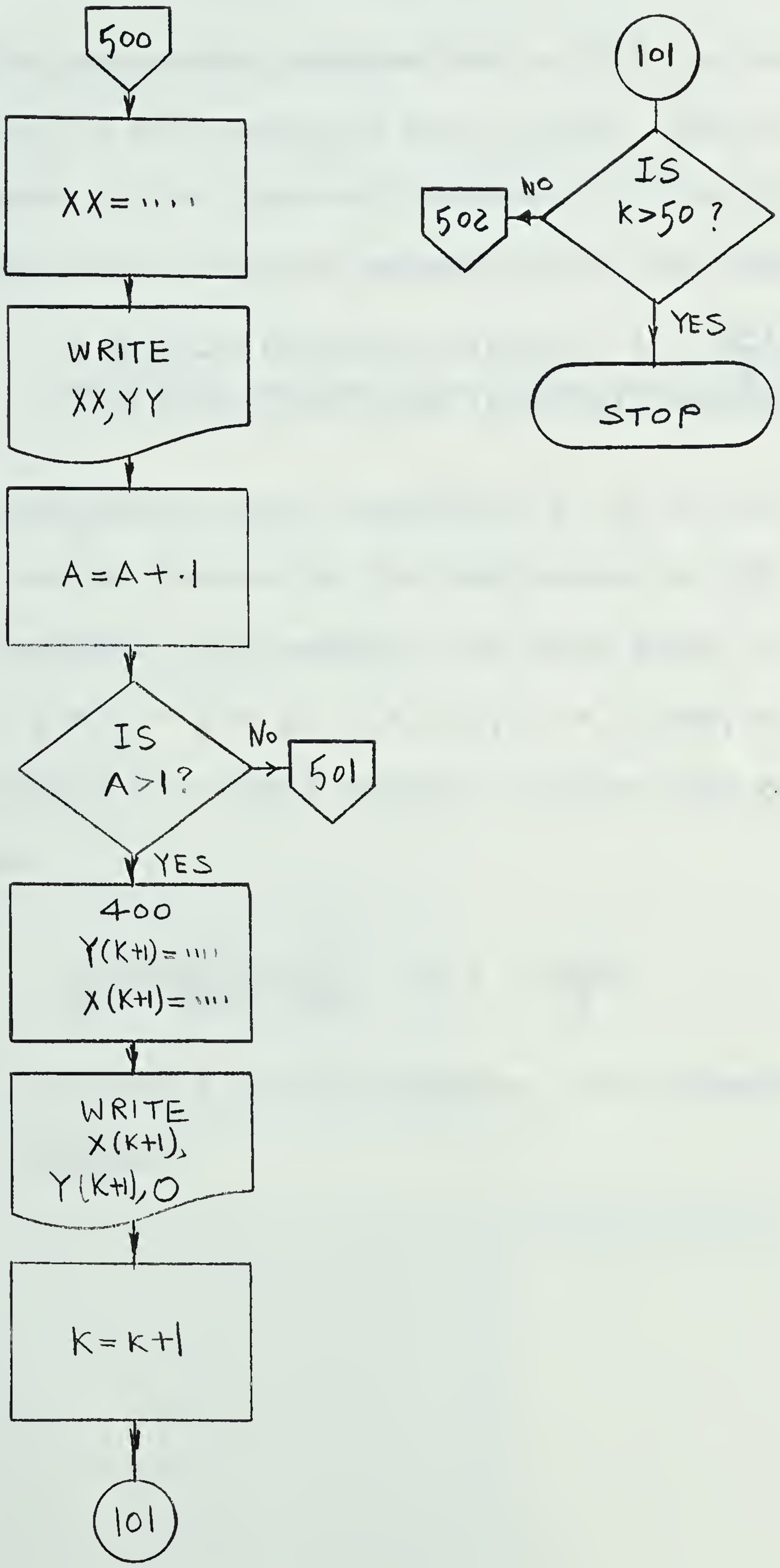
Throughout this work, several digital programs were written. They are intended as tools only. These are the following:

(1) Digital program for transient response.

Using equations derived from the state-variable method, No. 1, 2 of section 3, the transient response of the autonomous system, subject to initial conditions, was manipulated by the digital computer. Extension of this case to one with any input can be easily obtained by slight modification of the program. In the actual program, using computer variables, initial conditions are arbitrarily chosen to be $x = 3$, $y = 2$; with a gain of $C = 0.6$. Nine intersampling values are calculated. The complete program is shown as follows;

```
345008 CHENG                                    FORTRAN SOURCE LIST
      ISN          SOURCE STATEMENT

       0   $IBFTC S-T      NODECK
       1         DIMENSION X(200).Y(200)
       2         C=.6
       3         T=1.
       4         X(1)=3.
       5         Y(1)=2.
       6         K=1
       7       7 H=-X(K)
      10         IF (H.LT.0.) GO TO 4
      13         M=H+.5
      14         GO TO 5
      15       4 M=H-.5
      16       5 O=M
      17     300 U=O/H
      20         A=.1
      21       6 YY=-U*C*(1.-EXP(-A))* X(K)+Y(K)/EXP(A)
      22         XX=(1.-U*C*(A-1.+EXP(-A)))*X(K)+(1.-EXP(-A))*Y(K)
      23         WRITE (6,900) XX,YY
      24         A=A+.1
      25         IF (A.GT.1.) GO TO 400
      30         GO TO 6
      31     400 Y(K+1)=-U*C*(1.-EXP(-T))*X(K)+Y(K)/EXP(T)
      32         X(K+1)=(1.-U*C*(T-1.+EXP(-T)))*X(K)+(1.-EXP(-T))
                 *Y(K)
      33         WRITE (6,900) X(K+1), Y(K+1), O
      34         K=K+1
      35         IF (K.GT.50) CALL EXIT
      40         GO TO 7
      41     900 FORMAT (1X,4E16.8)
      42         END
```


START
DIMENSIONS
GAIN=0.6
SAMPLING PERIOD=1
I.C. X(1)=3
I.C. Y(1)=2
K=1
7 H=-X(K)
502
100
100
IS H<0 ?
YES
NO
M=H+.5
5
4 M=H-.5
5 O=M
300 U=O/H
A=.1
6 YY=....
501
500


500
XX= ....
WRITE XX,YY
A=A+.1
IS A>1?
No
501
YES
400
Y(K+1)= ....
X(K+1)= ....
WRITE X(K+1), Y(K+1), O
K=K+1
101
101
IS K>50 ?
NO
502
YES
STOP

(2) <u>Finding the optimal q</u>

The following program deals with the search for an optimal q for Jury and Lee's case. The value of q is evaluated in the interval between 10^{-4} to 10^{3}. The linear system used is quite general with the form of;

$$F(Z) = \frac{G\ Z^{n}(Z-A)(Z-B)(Z-C)(Z-D)(1 + q\,\frac{Z-1}{Z})}{(Z-O)(Z-P)(Z-R)(Z-S)(Z-U+jV)(Z-U-jV)}$$

Various values of the real constant, G, n, A, B, C, D, O, P, R, S, U, V can be chosen at the beginning to fit the particular system. For example, in this case, G = 0.368, A = -0.717, B = C = D = 0, O = 1.0, P = 0.368, R = S = U = V = 0, therefore n = 1 was chosen, so that the general form becomes;

$$F(Z) = \frac{0.368(Z+0.717)}{(Z-1)\ (Z-0.368)}\ (1 + q\,\frac{Z-1}{Z})$$

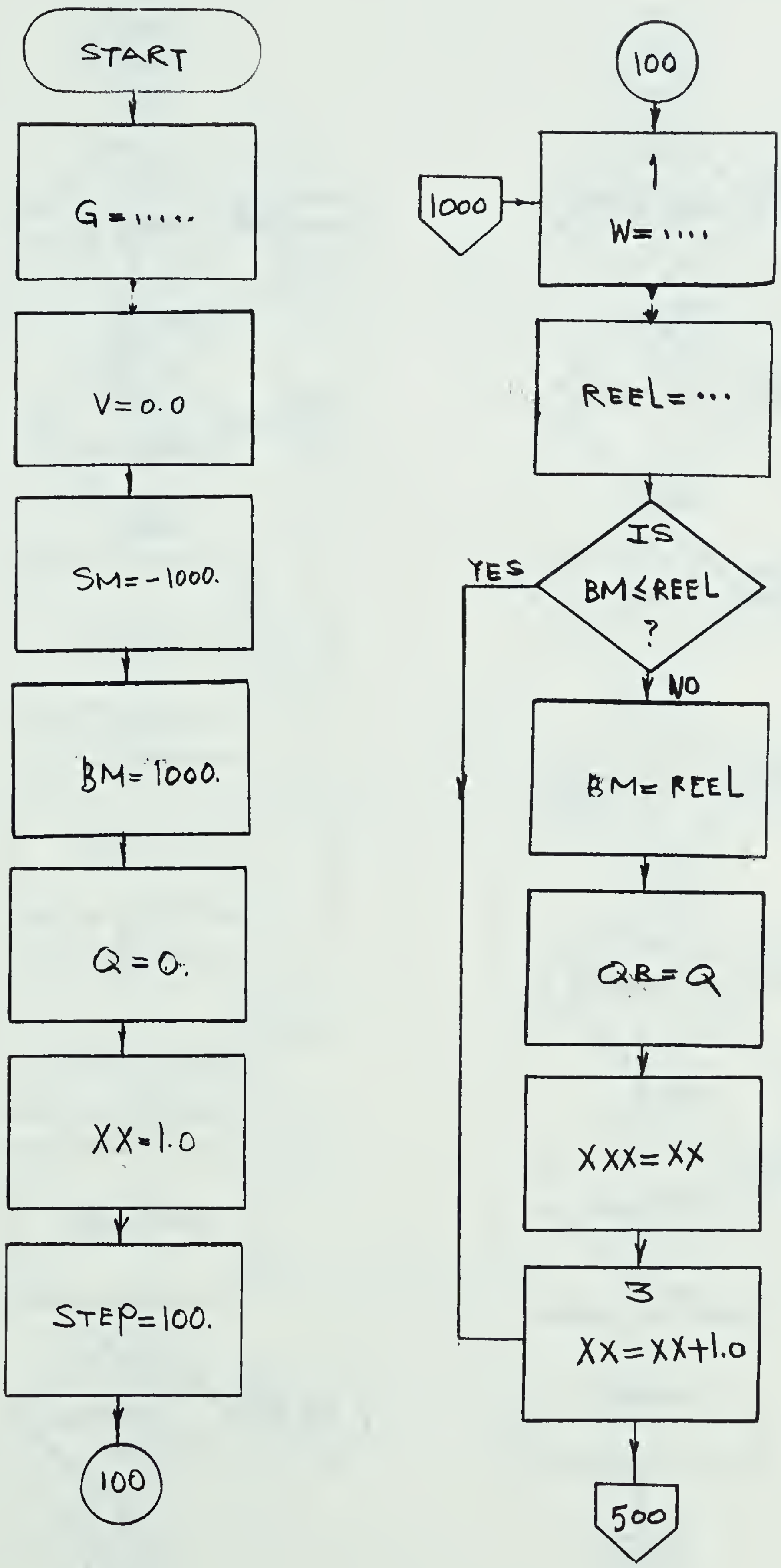
The index n becomes Z in the program. The complete program is as follows;

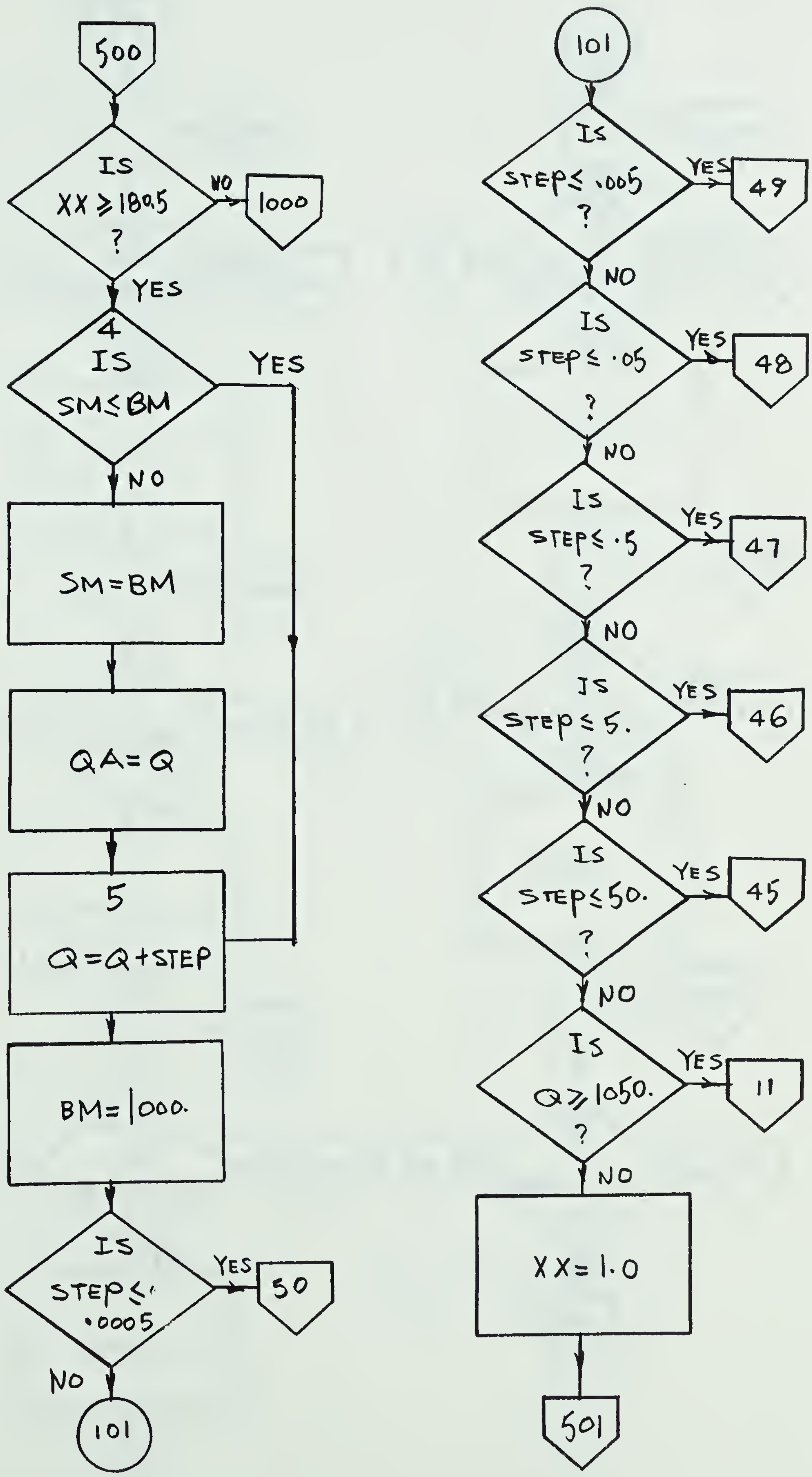
```
ISN              SOURCE STATEMENT

     $IBFTC CHENG   NOLIST,NODECK
            G=EXP(-1.)
            Z=1.
            A=(-1.)*(1.-2.*EXP(-1.))/EXP(-1.)
            B=.0
            C=.0
            D=.0
            O=1.0
            P=EXP(-1.)
            R=.0
            S=.0
            U=.0
            V=.0
            SM=-1000.
            BM=1000.
            Q=.0
            XX=1.0
            STEP=100.
          1 W=XX*.017453293
            X=COS(W)
            Y=SIN(W)
            AM=SQRT(X**2+Y**2)
            AMZ=AM**Z
            AMA=SQRT((X-A)**2+Y**2)
            AMB=SQRT((X-B)**2+Y**2)
            AMC=SQRT((X-C)**2+Y**2)
            AMD=SQRT((X-D)**2+Y**2)
            AMO=SQRT((X-O)**2+Y**2)
            AMP=SQRT((X-P)**2+Y**2)
            AMR=SQRT((X-R)**2+Y**2)
            AMS=SQRT((X-S)**2+Y**2)
            AMU=SQRT((X-U)**2+(V+Y)**2)
            AMV=AMU
            AMQ=(1.+Q)*SQRT((X-Q/(1.+Q))**2+Y**2)/AM
            SIZE=(G*AMZ*AMA*AMB*AMC*AMD*AMQ)/(AMO*AMP*AMR*AMS*AMU*AMV)
            ANZ=Z*57.29578*AT AN2(Y,X)
            ANA=57.29578*AT AN2(Y,X-A)
            ANB=57.29578*AT AN2(Y,X-B)
            ANC=57.29578*AT AN2(Y,X-C)
            ADD=57.29578*AT AN2(Y,X-D)
            ANO=57.29578*AT AN2(Y,X-O)
            ANP=57.29578*AT AN2(Y,X-P)
            ANR=57.29578*AT AN2(Y,X-R)
            ANS=57.29578*AT AN2(Y,X-S)
            ANU=57.29578*AT AN2(V+Y,X-U)
```

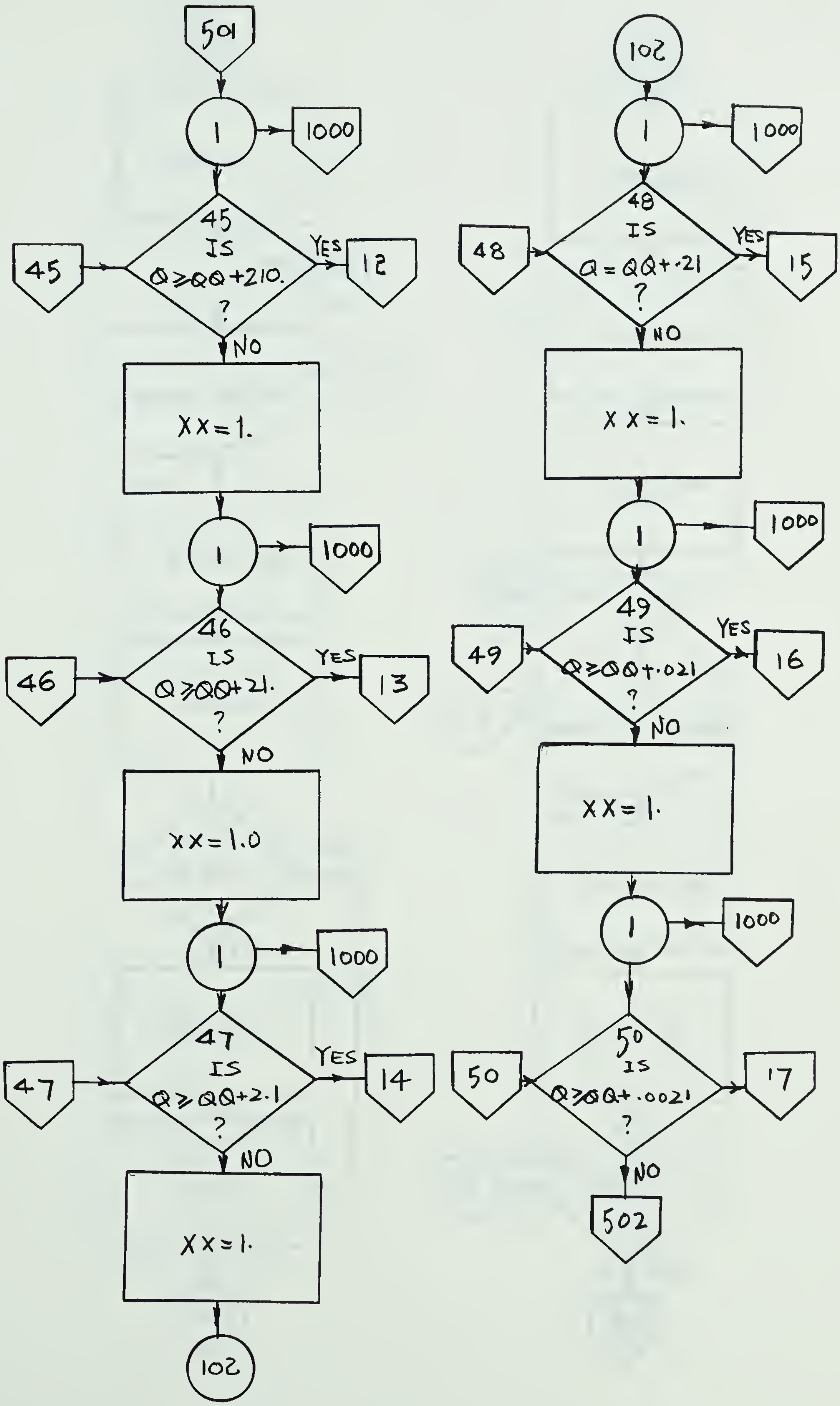
```
      ANV=57.29578*AT AN2(Y-V,X-U)
      ANQ=57.29578*(AT AN2(Y,(X-Q/(1.+Q)))-AT AN2(Y,X))
      PHASE=ANZ+ANA+ANB+ANC+ADD+ANQ-ANO-ANP-ANR-ANS-ANU-ANV
      REEL=SIZE*COS(PHASE*.017453293)
      IF (BM.LE.REEL) GO TO 3
      BM=REEL
      QB=Q
      XXX=XX
    3 XX=XX+1.0
      IF (XX.GE.180.5) GO TO 4
      GO TO 1
    4 IF (SM.GE.BM) GO TO 5
      SM=BM
      QA=Q
    5 Q=Q+STEP
      BM=1000.
      IF (STEP.LE.0.0005) GO TO 50
      IF (STEP.LE.0.005) GO TO 49
      IF (STEP.LE.0.05) GO TO 48
      IF (STEP.LE.0.5) GO TO 47
      IF (STEP.LE.5.0) GO TO 46
      IF (STEP.LE.50.) GO TO 45
      IF (Q.GE.1050.) GO TO 11
      XX=1.0
      GO TO 1
   45 IF (Q.GE.QQ+210.) GO TO 12
      XX=1.0
      GO TO 1
   46 IF (Q.GE.QQ+21.) GO TO 13
      XX=1.0
      GO TO 1
   47 IF (Q.GE.QQ+2.1) GO TO 14
      XX=1.0
      GO TO 1
   48 IF (Q.GE.QQ+.21) GO TO 15
      XX=1.0
      GO TO 1
   49 IF (Q.GE.QQ+.021) GO TO 16
      XX=1.0
      GO TO 1
   50 IF (Q.GE.QQ+.0021) GO TO 17
      XX=1.0
      GO TO 1
   11 WRITE (6,100) QA,SM
      Q=QA-100.
      QQ=Q
      IF (Q.GE.0.0) GO TO 30
      Q=0.0
   30 XX=1.
```

```
      STEP=10.
      SM=-1000.
      BM=1000.
      GO TO 1
   12 WRITE (6,100) QA,SM
      Q=QA-10.
      QQ=Q
      IF (Q.GE.0.0) GO TO 31
      Q=0.0
   31 XX=1.
      STEP=1.
      SM=-1000.
      BM=1000.
      GO TO 1
   13 WRITE (6,100) QA,SM
      Q=QA-1.
      QQ=Q
      IF (Q.GE.0.0) GO TO 32
      Q=0.0
   32 XX=1.
      STEP=.1
      SM=-1000.
      BM=1000.
      GO TO 1
   14 WRITE (6,100) QA,SM
      Q=QA-.1
      QQ=Q
      IF (Q.GE.0.0) GO TO 33
      Q=0.0
   33 XX=1.
      STEP=.01
      SM=-1000.
      BM=1000.
      GO TO 1
   15 WRITE (6,100) QA,SM
      Q=QA-.01
      QQ=Q
      IF (Q.GE.0.0) GO TO 34
      Q=0.0
   34 XX=1.
      STEP=.001
      SM=-1000.
      BM=1000.
      GO TO 1
   16 WRITE (6,100) QA,SM
      Q=QA-.001
      QQ=Q
      IF (Q.GE.0.0) GO TO 35
      Q=0.0
   35 XX=1.
```

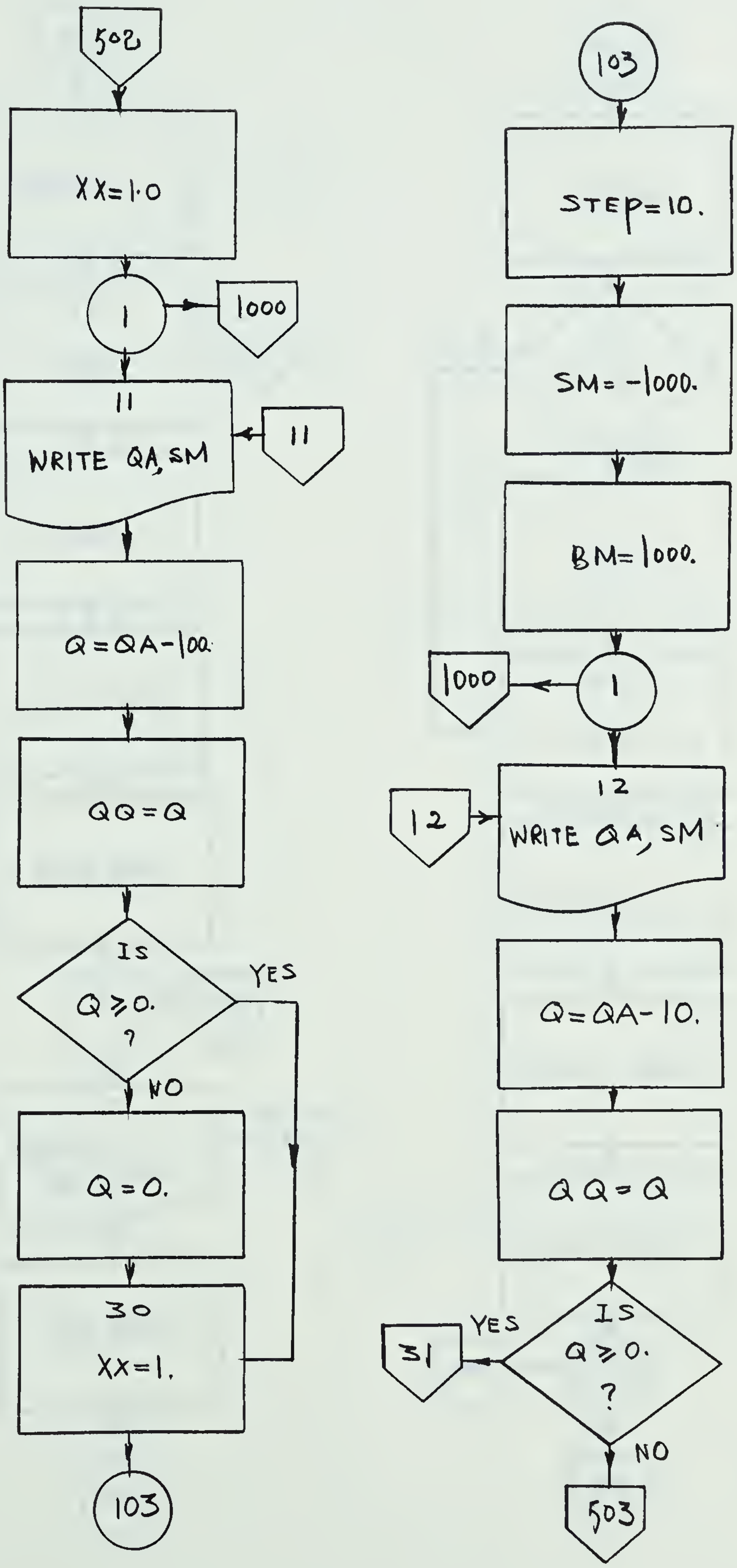
```
      STEP=.0001
      SM=-1000.
      BM=1000.
      GO TO 1
   17 WRITE (6,100) SM,QA
  100 FORMAT (2E20.8)
      CALL EXIT
      END
```

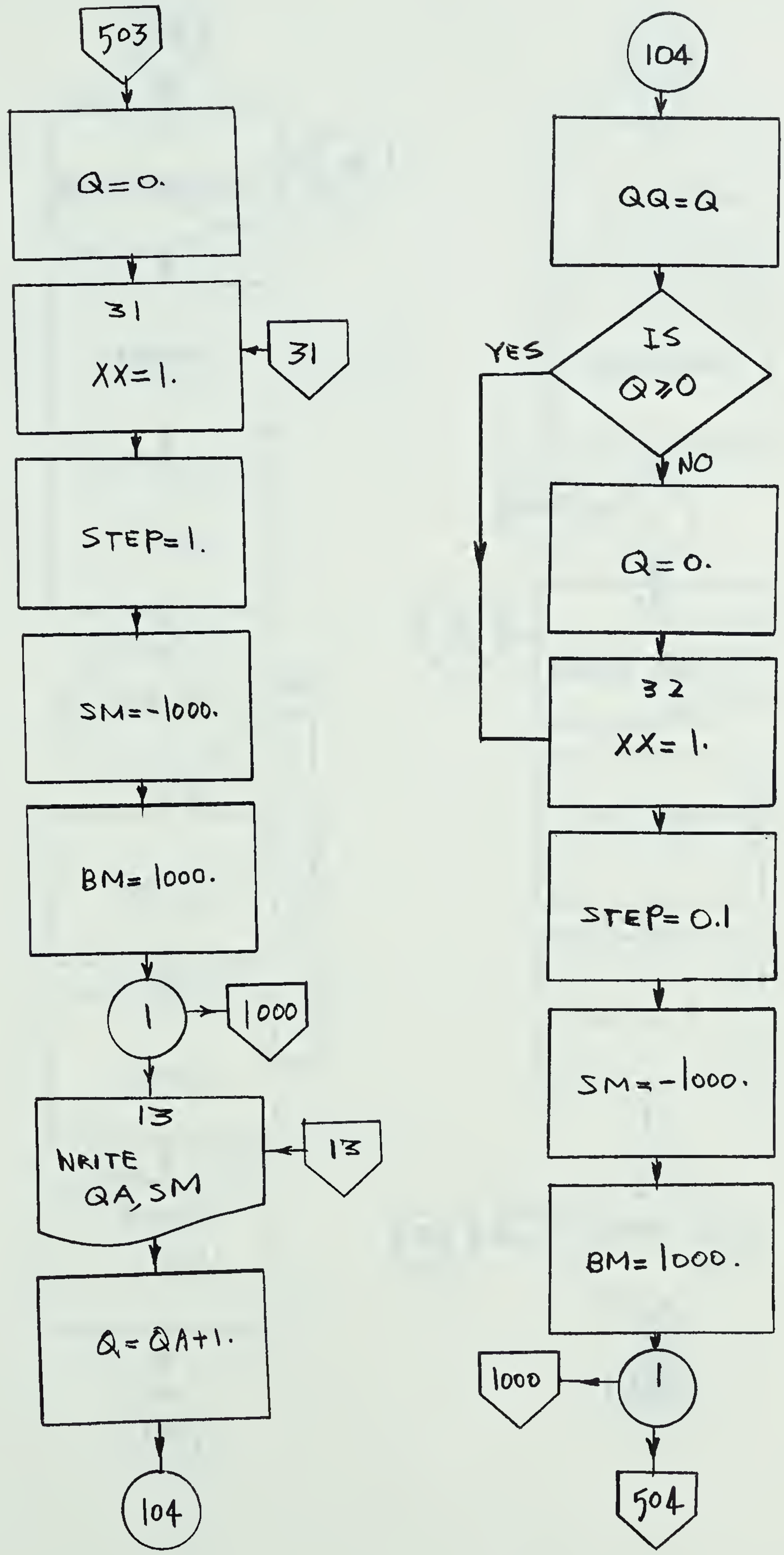

START
G = ...
V = 0.0
SM = -1000.
BM = 1000.
Q = 0.
XX = 1.0
STEP = 100.
100
100
1000
1
W = ...
REEL = . . .
IS
BM ≤ REEL
?
YES
NO
BM = REEL
QB = Q
XXX = XX
3
XX = XX+1.0
500

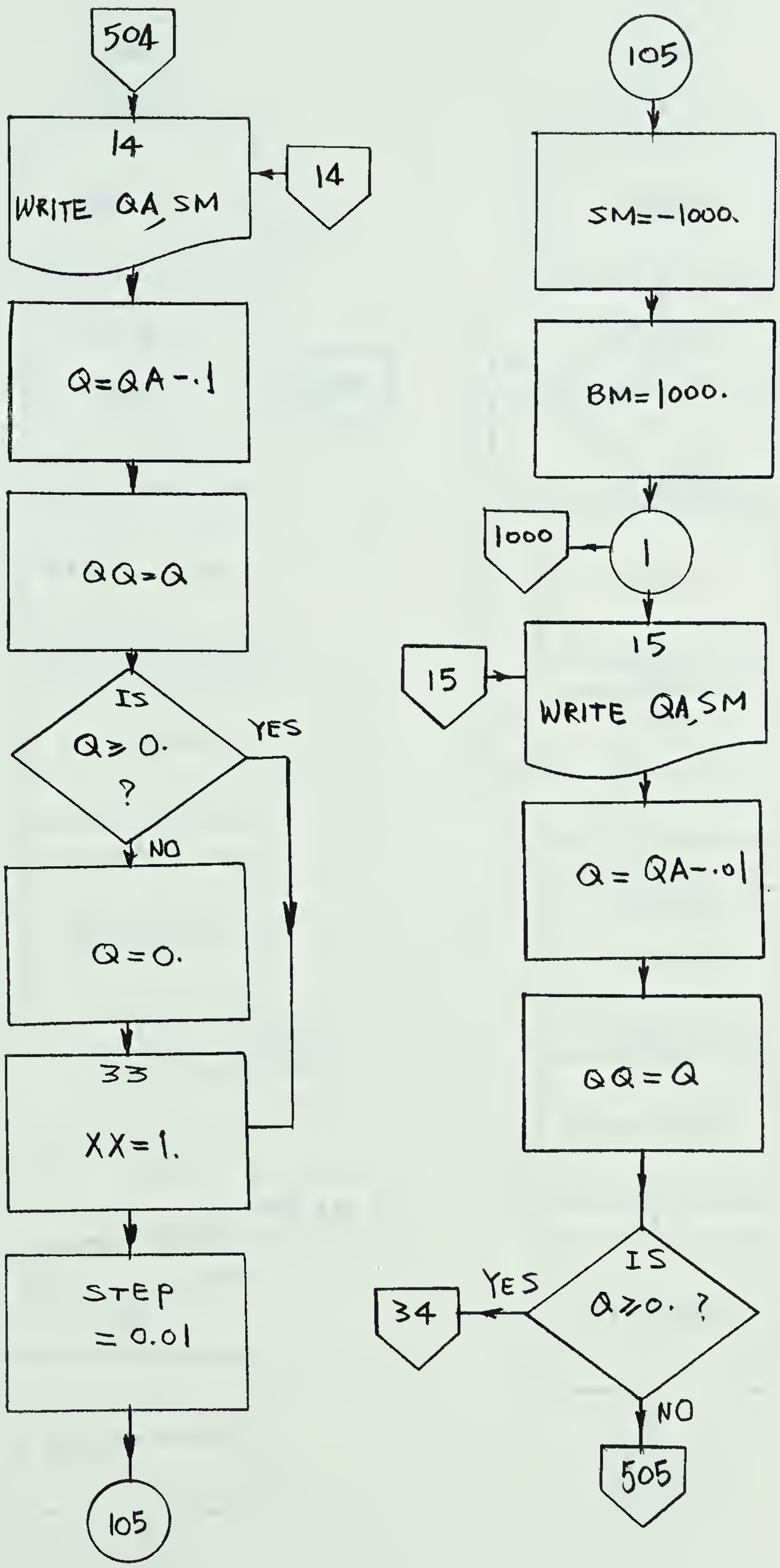

500
IS
XX ≥ 180.5
?
NO
1000
YES
4
IS
SM≤BM
YES
NO
SM=BM
QA=Q
5
Q=Q+STEP
BM=1000.
IS
STEP≤
.0005
YES
50
NO
101
101
IS
STEP≤ .005
?
YES
49
NO
IS
STEP ≤ .05
?
YES
48
NO
IS
STEP≤ .5
?
YES
47
NO
IS
STEP ≤ 5.
?
YES
46
NO
IS
STEP≤50.
?
YES
45
NO
IS
Q≥1050.
?
YES
11
NO
XX=1.0
501



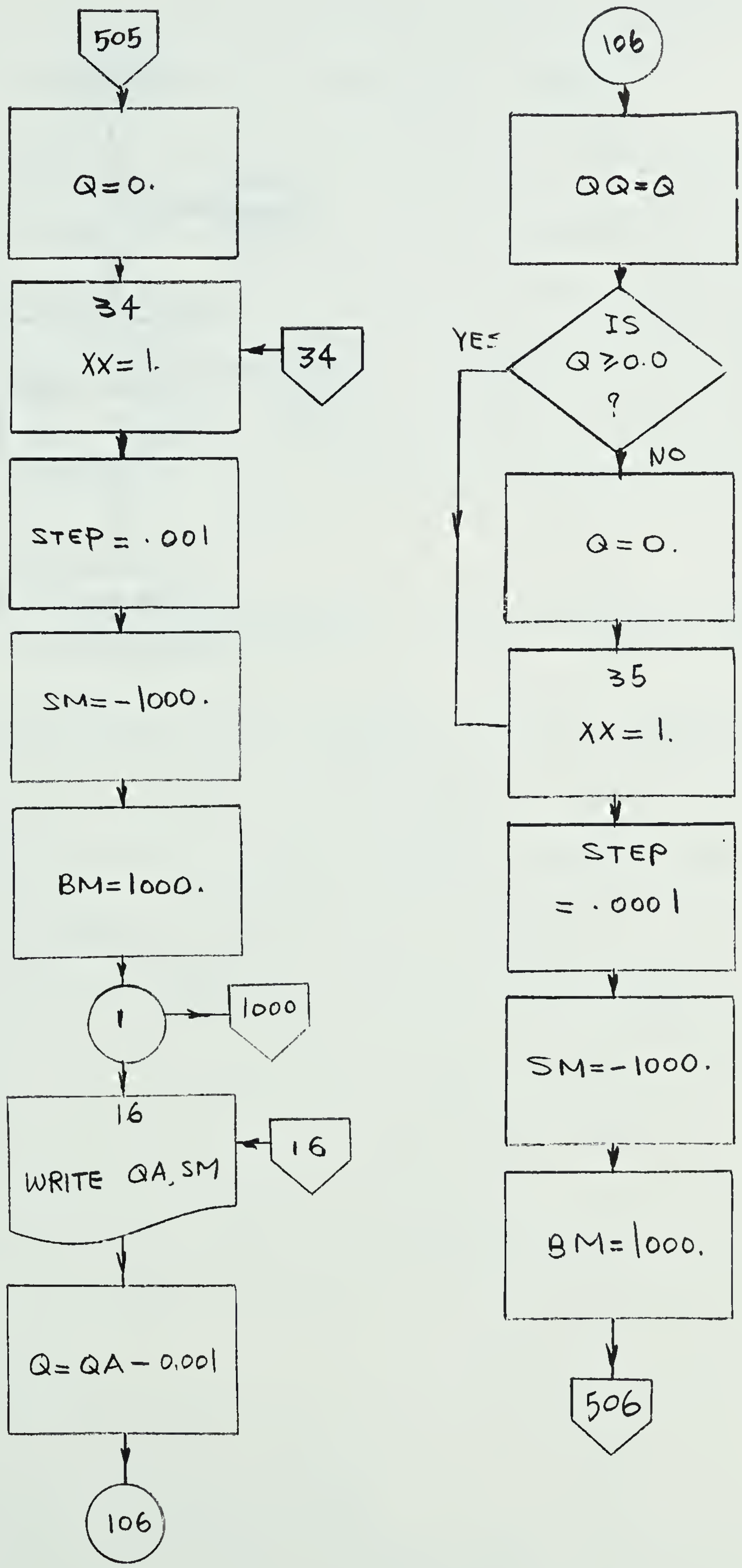
501
1
1000
45
45
IS
Q≥QQ+210.
?
YES
12
NO
XX=1.
1
1000
46
46
IS
Q≥QQ+21.
?
YES
13
NO
XX=1.0
1
1000
47
47
IS
Q≥QQ+2.1
?
YES
14
NO
XX=1.
102
102
1
1000
48
48
IS
Q=QQ+.21
?
YES
15
NO
XX=1.
1
1000
49
49
IS
Q≥QQ+.021
?
YES
16
NO
XX=1.
1
1000
50
50
IS
Q≥QQ+.0021
?
17
NO
502

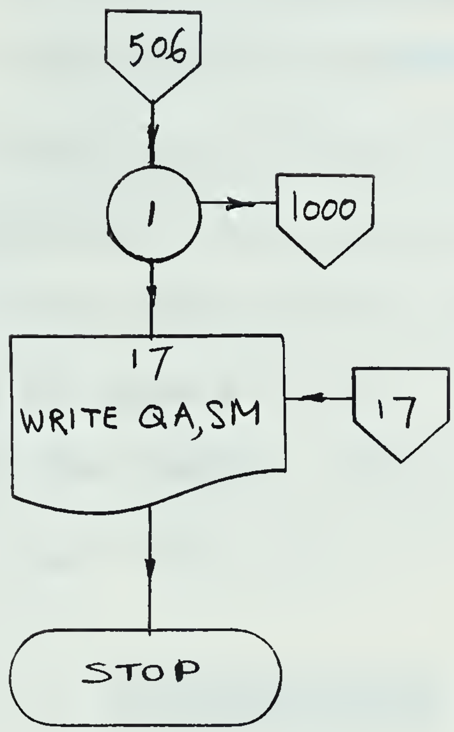

502
XX=1.0
1
1000
11
WRITE QA,SM
11
Q=QA-100.
QQ=Q
IS
Q≥0.
?
YES
NO
Q=0.
30
XX=1.
103
103
STEP=10.
SM=-1000.
BM=1000.
1000
1
12
12
WRITE QA,SM
Q=QA-10.
QQ=Q
YES
IS
Q≥0.
?
31
NO
503


503
Q=0.
31
XX=1.
31
STEP=1.
SM=-1000.
BM=1000.
1
1000
13
WRITE
QA, SM
13
Q=QA+1.
104
104
QQ=Q
IS
Q≥0
YES
NO
Q=0.
32
XX=1.
STEP=0.1
SM=-1000.
BM=1000.
1000
1
504



504
14
WRITE QA, SM
14
Q=QA-.1
QQ=Q
IS
Q≥0.
?
YES
NO
Q=0.
33
XX=1.
STEP
=0.01
105
105
SM=-1000.
BM=1000.
1000
1
15
15
WRITE QA, SM
Q = QA-.01
QQ=Q
IS
Q≥0. ?
YES
34
NO
505


505
Q=0.
34
XX=1.
34
STEP=.001
SM=-1000.
BM=1000.
1
1000
16
WRITE QA, SM
16
Q=QA-0.001
106
106
QQ=Q
IS
Q≥0.0
?
YES
NO
Q=0.
35
XX=1.
STEP
=.0001
SM=-1000.
BM=1000.
506


506
1
1000
17
WRITE QA,SM
17
STOP

(3) <u>To plot the gain-phase diagram and to calculate the minimum real part.</u>

After the optimal q is found, the following program is concerned with obtaining information to enable plotting of the compensated system in the gain-phase plot as Z takes on values on $|Z| = 1$ from $\omega T = 1°$ to $\omega T = 180°$ in steps of one degree. This program is meant for $q = 0.87$, $\omega_m = 2.05$, $\phi_m = 30°$, or

$$F_c(Z) = \frac{0.368(Z+0.717)}{(Z-1)(Z-0.368)} \left(1 + 0.87\,\frac{Z-1}{Z}\right)\left(\frac{Z-B}{Z-A}\right)$$

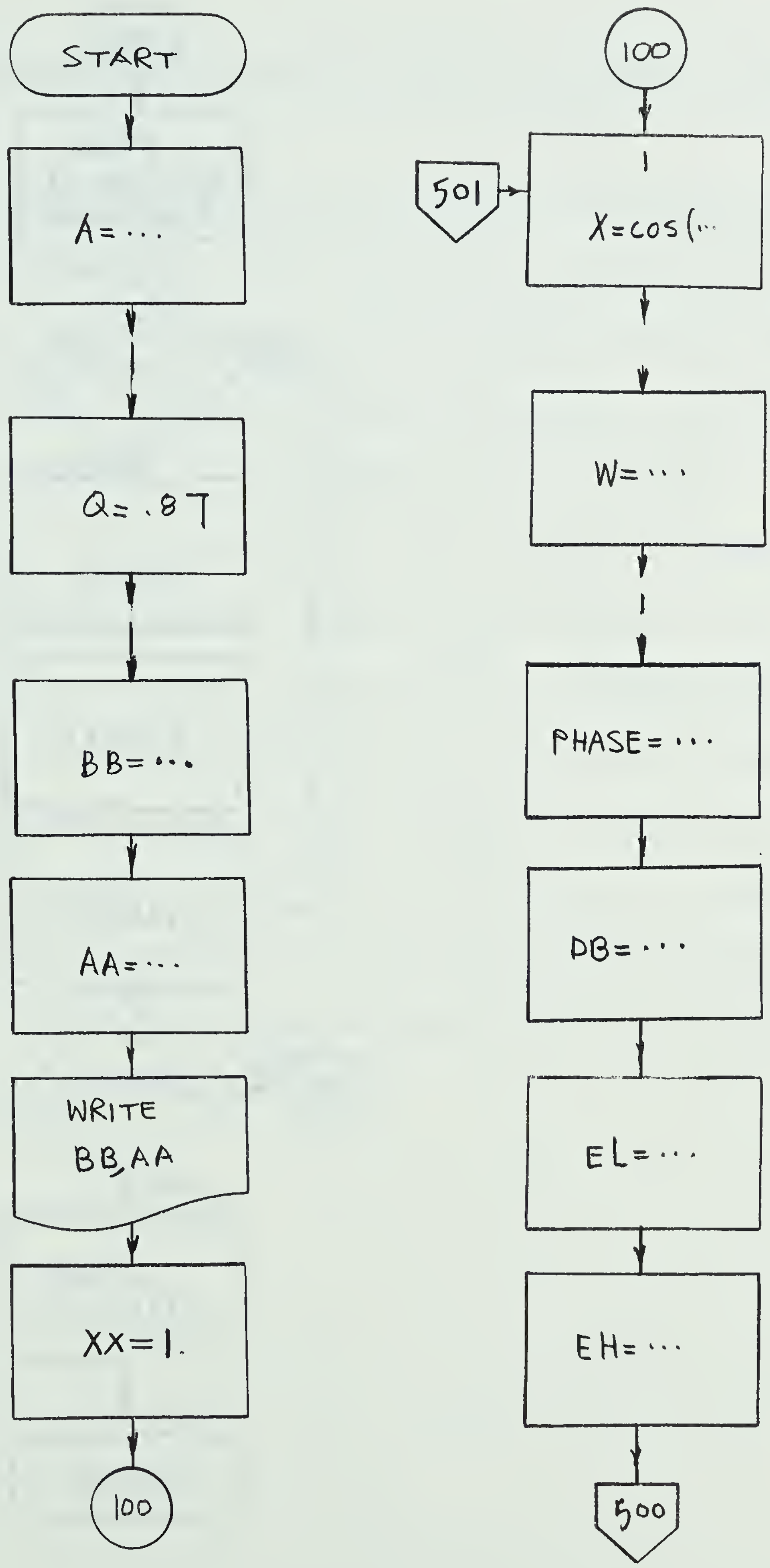
where the expressions for finding B, A as determined by the choice of ω_m and ϕ_m is included in the program.

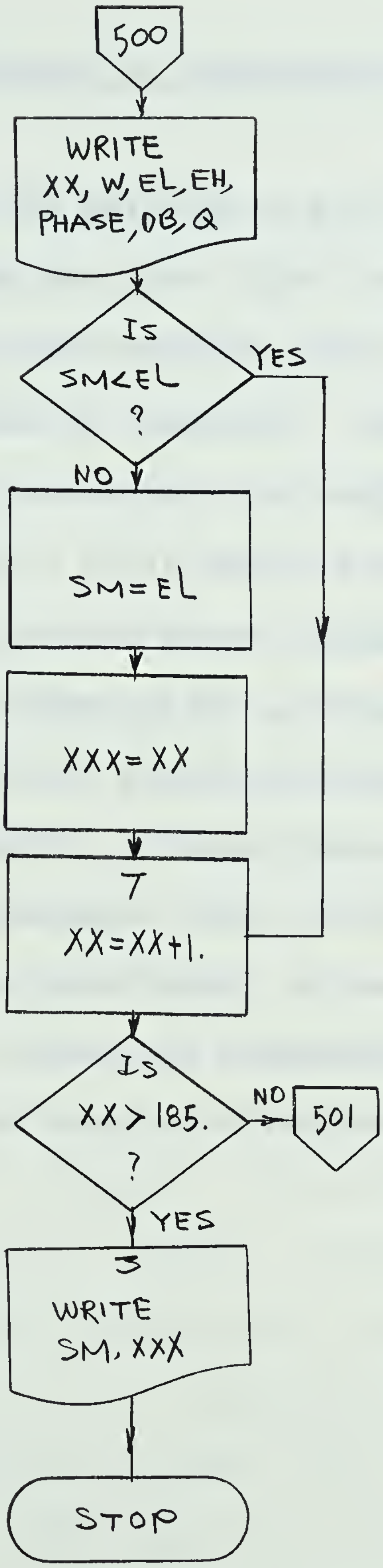
Notation q, B, A, ω_m, ϕ_m become respectively in the computer variables, Q, BB, AA, WM, FIM.

The complete program is given as follows;

345008 FORTRAN SOURCE LIST

```
ISN                    SOURCE STATEMENT

  0  $IBFTC CHENG   NOLIST, NODECK
  1        A=(1.-2.*EXP(-1.))/EXP(-1.)
  2        B=EXP(-1.)
  3        Q=.87
  4        H=1.
  5        SM=1000.
  6        T=1.0
  7        WM=2.05
 10        FIM=30.*0.017453293
 11        BB=(1.+SIN(FIM-WM*T))/(SIN(FIM)+COS(WM*T))
 12        AA=(SIN(FIM)-COS(WM*T))/(-1.+SIN(FIM-WM*T))
 13        WRITE (6,200) BB,AA
 14    200 FORMAT (2E20.8)
 15        XX=1.
 16      1 X=COS(XX*.017453293)
 17        Y=SIN(XX*.017453293)
 20        W=XX*.017453293
 21        AMA=SQRT((X+A)**2+Y**2)
 22        AMB=SQRT((X-1.)**2+Y**2)
 23        AMC=SQRT((X-B)**2+Y**2)
 24        ANA=57.29578*AT AN2(Y,X+A)
 25        ANB=57.29578*AT AN2(Y,X-1.)
 26        ANC=57.29578*AT AN2(Y,X-B)
 27        AMQ=SQRT(((1.+Q)*X-Q)**2+((1.+Q)*Y)**2)
 30        AMZ=SQRT(X**2+Y**2)
 31        ANQ=57.29578*AT AN2((1.+Q)*Y,(1.+Q)*X-Q)
 32        ANZ=57.29578*AT AN2(Y,X)
 33        SIZ=SQRT((X-BB)**2+Y**2)/SQRT((X-AA)**2+Y**2)
 34        PD=57.29578*(AT AN2(Y°X-BB)-AT AN2(Y,X-AA))
 35        AMG=(H*B*AMA*AMQ*SIZ)/(AMB*AMC*AMZ)
 36        PHASE=ANA-ANB-ANC+PD+ANQ-ANZ
 37        DB=20.*ALOG10(AMG)
 40        EL=AMG*COS(PHASE*.017453293)
 41        EH=AMG*SIN(PHASE*.017453293)
 42        WRITE (6,100) XX,W,El,EH,PHASE,DB,Q
 43        IF (SM.LT.EL) GO TO 7
 46        SM=EL
 47        XXX=XX
 50      7 XX=XX+1
 51        IF (XX.GT.185.) GO TO 3
 54        GO TO 1
 55      3 WRITE (6,105) SM,XXX
 56    100 FORMAT (7E16.4)
 57   105  FORMAT(/5X,22HTHE MINIMUM REAL PART,E12.4,22H,OCCURS AT
           AN ANGLE 0 1F,E15.4,9H DEGREES.///)
 60        CALL EXIT
 61        END
```


START
A= ...
Q= .8T
BB= ...
AA= ...
WRITE
BB,AA
XX=1.
100
100
501
X=cos(...
W= ...
PHASE= ...
DB= ...
EL= ...
EH= ...
500


500
WRITE
XX, W, EL, EH, PHASE, DB, Q
IS
SM<EL
?
YES
NO
SM=EL
XXX=XX
7
XX=XX+1.
IS
XX>185.
?
NO
501
YES
3
WRITE
SM, XXX
STOP

3 Discussion of accuracy of the graphical method

When the gain-phase plot of the plant has a very steep slope near the right end of the graph, difficulty of matching the tangent point of the two curves results in a problem of accuracy. This difficulty can be decreased by redrawing the graphs with an expanded phase angle scale. It is best to use a digital computer in order to find the exact minimum real value of the plant. As far as compensation is concerned, a proper filter can be found by the graphical method so that the tangent point occurs at a higher phase angle. With the aid of a digital computer there is no difficulty in determining whether the requirement of the compensated system is met or not after the compensating network has been found by the graphical method.

10 References

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